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Two integral presumptions

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Preface

The following two formulae are stated:

First integral presumption: $I_1(p) = \int_{-\pi}^{\pi} \frac{\sin^2 px}{\sin^2 x} dx = |p| 2\pi$

Second integral presumption: $I_2(m,n) = \int_{-\pi}^{\pi} \frac{\sin mx \sin nx}{\sin^2 x} dx = \left(\left| \frac{m+n}{2} \right| - \left| \frac{m-n}{2} \right| \right) 2\pi$

The last part of this paper is an abstract of the original referred article by Andrew Antkiller.

Theoretical background

This work is based on the famous work of dr. Andrew Antkiller (Math. Proc. Trig. Calc. 21,4. Cardiff 1876). Among other formulae he stated the following:

For any arbitrary real number a the following relations are valid:

1. $\cos^2 a + \sin^2 a = 1$ hence by rearrangement $\cos^2 a = 1 - \sin^2 a$
2. $\sin(2a) = 2 \sin a \cos a$
3. $\cos(2a) = \cos^2 a - \sin^2 a$

Given two arbitrary real numbers a and b the following relations are true:

4. $\sin(a+b) = \sin a \cos b + \cos a \sin b$
5. $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$

Among other relations, theese are mentioned in the appendix.

First integral presumption

Let us prove the relation $I_1(p) = \int_{-\pi}^{\pi} \frac{\sin^2 px}{\sin^2 x} dx = |p| 2\pi$

Consider the integrand. Using Antkiller 4 yields

$$\frac{\sin^2 px}{\sin^2 x} = \frac{\sin^2(x+(p-1)x)}{\sin^2 x} = \frac{(\sin x \cos(p-1)x + \cos x \sin(p-1)x)^2}{\sin^2 x}$$

$$= \frac{\sin^2 x \cos^2(p-1)x + \cos^2 x \sin^2(p-1)x + 2\sin x \cos(p-1)x \cos x \sin(p-1)x}{\sin^2 x}$$

The three addends in the numerator yield

I: $\dfrac{\sin^2 x\cos^2(p-1)x}{\sin^2 x} = \cos^2(p-1)x$

II: $\dfrac{\cos^2 x\sin^2(p-1)x}{\sin^2 x} = \dfrac{(1-\sin^2 x)\sin^2(p-1)x}{\sin^2 x} = \dfrac{\sin^2(p-1)x}{\sin^2 x} - \sin^2(p-1)x$

III: $\dfrac{2\sin x\cos(p-1)x\cos x\sin(p-1)x}{\sin^2 x} = \dfrac{\sin 2x\sin 2(p-1)x}{2\sin^2 x}$

In III we now apply the method henceforth known as the Amtrup-Antkiller Algorithm:

Applying Antkiller 5 with $a+b = 2(p-1)x$ and $a-b = 2x$ i.e. $a = px$ and $b = (p-2)x$ we have

III: $\dfrac{\sin^2 px - \sin^2(p-2)x}{\sin^2 x} = \dfrac{\sin^2 px}{\sin^2 x} - \dfrac{\sin^2(p-2)x}{\sin^2 x}$

Adding theese three contributions (consisting of five addends) we have

$$\frac{\sin^2 px}{\sin^2 x} = \cos^2(p-1)x + \frac{\sin^2(p-1)x}{\sin^2 x} - \sin^2(p-1)x + \frac{\sin^2 px}{2\sin^2 x} - \frac{\sin^2(p-2)x}{2\sin^2 x}$$
$$= \frac{\sin^2(p-1)x}{\sin^2 x} + \frac{\sin^2 px}{2\sin^2 x} - \frac{\sin^2(p-2)x}{2\sin^2 x} + \cos 2(p-1)x$$

Integrating this from $-\pi$ to π yields

$I_1(p) = I_1(p-1) + ½I_1(p) - ½I_1(p-2) + 0$ for $p \neq 1$ and

$I_1(p) = I_1(p-1) + ½I_1(p) - ½I_1(p-2) + 2\pi$ for $p = 1$

So we have for $p \neq 1$

$½I_1(p) = I_1(p-1) - ½I_1(p-2)$ or $I_1(p) = 2I_1(p-1) - I_1(p-2)$

And finally

$I_1(p) - I_1(p-1) = I_1(p-1) - I_1(p-2)$ independent of p.

Likewise, for $p = 1$ we have

$I_1(p) - I_1(p-1) = I_1(p-1) - I_1(p-2) + 4\pi$

Recursion leads to

$$I_1(p) - I_1(p-1) = I_1(1) - I_1(0) = \int_{-\pi}^{\pi}\frac{\sin^2 x}{\sin^2 x}dx - \int_{-\pi}^{\pi}0dx = 2\pi \quad \text{for } p > 1 \quad \text{and}$$

$$I_1(p) - I_1(p-1) = I_1(0) - I_1(-1) = \int_{-\pi}^{\pi}0dx - \int_{-\pi}^{\pi}\frac{\sin^2(-x)}{\sin^2 x}dx = -2\pi \quad \text{for } p < 1$$

Therefore

$I_1(p) = I_1(0) + |p|2\pi$ or

$$\int_{-\pi}^{\pi}\frac{\sin^2 px}{\sin^2 x}dx = |p|2\pi$$

which veryfies the first integral presumption.

Corrolary

It can now be veryfied, that the integral

$$\int_{-\pi}^{\pi} \frac{2\sin x \cos(p-1)x \cos x \sin(p-1)x}{\sin^2 x} \mathrm{d}x$$

Has the value 2π for $p>1$, 0 for $p=1$ and -2π for $p<1$.

Second integral presumption

Now we will prove the relation $I_2(m,n) = \int_{-\pi}^{\pi} \frac{\sin mx \sin nx}{\sin^2 x} dx = \left(\left|\frac{m+n}{2}\right| - \left|\frac{m-n}{2}\right|\right) 2\pi$

This may be veryfied by applying, again, the Amtrup-Antkiller Algorithm. Using Antkiller 5 with

$a+b = mx$ and $a-b = nx$ i.e. $a = \frac{m+n}{2}x$ and $b = \frac{m-n}{2}x$ yields

$$\frac{\sin mx \sin nx}{\sin^2 x} = \frac{\sin^2\left(\frac{m+n}{2}\right)x - \sin^2\left(\frac{m-n}{2}\right)x}{\sin^2 x}$$

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Kræva . $\frac{m+n}{2}$, $\frac{m-n}{2}$ $\in \mathbb{Z}$!

Hence

$$I_2(m,n) = I_1\left(\frac{m+n}{2}\right) - I_1\left(\frac{m-n}{2}\right) = \left(\left|\frac{m+n}{2}\right| - \left|\frac{m-n}{2}\right|\right) 2\pi$$

which is the second integral presumption.

If $m,n \in \mathbf{R}_+$ and $m>n$ we have

$$\int_{-\pi}^{\pi} \frac{\sin mx \sin nx}{\sin^2 x} dx = \left(\frac{m+n}{2} - \frac{m-n}{2}\right) 2\pi = n2\pi$$

m-n lige

where n is the least of the two numbers m and n.

ellers 0

Appendix: Summary of the Antkiller reductions

Of the first set of formulae, the first is proven by unit-vector calculus and the others by changing the sign of v respectively replacing u by $\pi/2 - u$. In all formulae u, v, a and b are all arbitrary real numbers.

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

Setting $u = v$ above yield

$0 = 2 \sin \tfrac{1}{2}(x-y) \sin \tfrac{1}{2}(x-z) \sin \tfrac{1}{2}(y-z)$

$= 2(\sin(x-y) + \sin(y-z) + \sin(z-x))$

$$1 = \cos^2 u + \sin^2 u$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

$$0 = 0$$

$$\sin 2u = 2 \sin u \cos u$$

Adding and subtracting pairs of formulae above yield

$$\cos(u+v) + \cos(u-v) = 2 \cos u \cos v$$

$$\cos(u+v) - \cos(u-v) = -2 \sin u \sin v$$

$$\sin(u+v) + \sin(u-v) = 2 \sin u \cos v$$

$$\sin(u+v) - \sin(u-v) = 2 \cos u \sin v$$

Setting $u+v=a$ and $u-v=b$ i.e. $u = \dfrac{a+b}{2}$ and $v = \dfrac{a-b}{2}$ we have

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$$

Deviding by two and replacing a with $2a$ and b with $2b$ we get

$$\cos(a+b)\cos(a-b) = \tfrac{1}{2}\cos 2a + \tfrac{1}{2}\cos 2b = \cos^2 a + \cos^2 b - 1 = 1 - \sin^2 a - \sin^2 b$$

$$\sin(a+b)\sin(a-b) = -\tfrac{1}{2}\cos 2a + \tfrac{1}{2}\cos 2b = \cos^2 b - \cos^2 a = \sin^2 a - \sin^2 b$$

$$\sin(a+b)\cos(a-b) = \tfrac{1}{2}\sin 2a + \tfrac{1}{2}\sin 2b = \sin a \cos a + \sin b \cos b$$

$$\cos(a+b)\sin(a-b) = \tfrac{1}{2}\sin 2a - \tfrac{1}{2}\sin 2b = \sin a \cos a - \sin b \cos b$$

The second formula above is the Antkiller 5 formula. Adding and subtracting pairs of formulae we have

$$\cos(a+b)\cos(a-b) + \sin(a+b)\sin(a-b) = 2\cos^2 b - 1 = \cos 2b$$

$$\cos(a+b)\cos(a-b) - \sin(a+b)\sin(a-b) = 2\cos^2 a - 1 = \cos 2a$$

$$\sin(a+b)\cos(a-b) + \cos(a+b)\sin(a-b) = 2 \sin a \cos a = \sin 2a$$

$$\sin(a+b)\cos(a-b) - \cos(a+b)\sin(a-b) = 2 \sin b \cos b = \sin 2b$$

Note the independence of one of the variables a or b. The next step is to replace $a+b$ with u and $a-b$ with v. Obtaining theese interesting results is left to the reader. It will verify, that any set of four formulae is equivalent to any other set.

$\cos px \cos ry - \cos rx \cos py = \sin nu \sin mt + \sin mu \sin nt$ $\quad n = p+r$, $m = r-p$ $\quad u = \dfrac{x+y}{2}$, $t = \dfrac{x-y}{2}$

$\cos px \sin qy - \sin qx \cos py = \cos nu \sin mt - \cos mu \sin nt$ $\quad n = p+q$, $m = p-q$

$\sin qx \cos py - \cos px \sin qy = \cos mu \sin nt - \cos nu \sin mt$

$\sin qx \sin ry - \sin rx \sin qy = \cos(r+q)u \cos(r-q)t - \cos(r-q)u \cos(r+q)t$