Spinning Strings in Lunin-Maldacena Background and Their Gauge Theory Duals

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Abstract

In this master thesis we calculate the classical energy of circular strings with three independent angular momenta in the gravity background found by Lunin and Maldacena. For large angular momenta, quantum corrections are suppressed and the result is shown to agree with the conformal dimension of $SU(3)$ bosonic operators in β-deformed $\mathcal{N}=4$ SYM, as predicted by the AdS/CFT correspondence.

The approach to the subject will be introductory and we thus start with a brief introduction to the Maldacena conjecture, anti-de Sitter space, and conformal field theory. The basic properties of $\mathcal{N}=4$ SYM is then reviewed and it is shown that calculating anomalous dimensions in the planar limit amounts to diagonalizing the hamiltonian of an integrable spin chain. The coordinate Bethe ansatz is introduced and used to derive the eigenvalues of the hamiltonian in the thermodynamic limit. The formalism is first developed for the $SU(2)$ spin chain, then generalized to the $SU(3)$ spin chain, and finally β-deformed spin chains. Hence, we obtain the one-loop anomalous dimensions of $SU(3)$ bosonic operators in β-deformed $\mathcal{N}=4$ SYM. We conclude by reviewing the deformation of the five-sphere conjectured to yield the gravity dual of β-deformed $\mathcal{N}=4$ SYM and calculate the energy of spinning strings in this background.

Contents

1 Introduction

String theory has long been regarded as a very promising candidate for a unified theory of all interactions. One of its many beauties is that it gives a natural explanation of gauge symmetry in terms of open strings and D-branes, and string theory can thus be viewed as a fundamental theory implying the existence of gauge theories. Curiously, the theory was discovered in an attempt to describe strongly interacting particles. However, it was soon replaced by QCD, and strings were then regarded as an effective description of the thin tubes of color flux lines mediating the interactions between quarks. The AdS/CFT correspondence is a duality that relates a certain gauge theory and a string theory in a particular gravity background. The two theories are conjectured to be equivalent and as such, the question of which is more fundamental becomes redundant.

The first proposal of a gauge theory/string duality was given in 1974 by 't Hooft. The idea is very general and is based on the similarities between the diagrammatic expansions of large N gauge theory and string theory. In the gauge theory, the diagrams can be classified according to their topology with $1/N$ counting the genus and the perturbation expansion can be reorganized as a genus expansion in $1/N$. The non-planar diagrams corresponding to string interactions are suppressed in the large N limit and a new effective coupling constant $\lambda \equiv N g_{YM}^2$ (the 't Hooft coupling) counts the quantum loops of the planar diagrams.

Since the work of 't Hooft, it has been widely suspected that such a duality should exist, but it has proven hard to actually find the string theory corresponding to a four-dimensional gauge theory. First of all, the planar diagram expansion that corresponds to the free string theory is very complicated. Furthermore, the construction of a string theory in four spacetime dimensions are plagued with a number of difficulties, and one is led to introduce at least one extra dimension.

A concrete proposal of a gauge/string duality was given by Maldacena in 1997 [1]. He conjectured that type IIB string theory, in a product of five-dimensional anti-de Sitter space and a five-dimensional sphere ($AdS_5 \times S^5$) with common radius of curvature R, is dual to a maximally supersymmetric Yang-Mills theory with gauge group $U(N)$ in four-dimensional Minkowski space ($\mathcal{N} = 4$ SYM). The gauge theory is a conformal field theory, and the duality has been dubbed the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. The gauge theory is characterized by the two parameters g_{YM} and N, which are the coupling constant and rank of gauge group, respectively, and the string theory is characterized by $R/\sqrt{\alpha'}$ and g_s, which are the radius of curvature (in string units) and string coupling, respectively. The correspondence predicts how these should be related:

$$\frac{R^4}{\alpha'^2} = g_{YM}^2 N = \lambda, \qquad 4\pi g_s N = g_{YM}^2 N = \lambda. \tag{1.1}$$

The 't Hooft limit of the gauge theory, where $N \to \infty$ with λ fixed, corresponds to the free string theory.

The correspondence realizes a long-standing suspicion that a theory of quantum gravity should have a holographic description [2, 3]. This means that a gravitational theory on a given spacetime should be described by some theory on its boundary. This is in part inspired by the physics of black holes, as the entropy of a black hole can be shown to be proportional to the area of the horizon rather than the volume enclosed by the horizon. Although anti-de Sitter space is not compact, there is a precise sense in which we can associate four-dimensional Minkowski space where the gauge theory lives with the boundary of AdS_5. Since the AdS/CFT correspondence states the equivalence of a non-gravitational gauge theory on the four-dimensional boundary of AdS_5 with a string theory containing gravity in the interior of AdS_5, the correspondence provides a holographic description of gravity.

A first indicator of the correspondence is the underlying symmetries in the two theories. Both are invariant under the superconformal group $PSU(2,2|4)$, which has the bosonic subgroup $SO(2,4) \times SO(6)$. However, the symmetries enter in a different manner in the two theories. The subgroup $SO(6)$, for instance, is an internal symmetry of the gauge theory, whereas it is a symmetry of spacetime in the string theory.

If the AdS/CFT correspondence is true, one should be able to match the conserved charges corresponding to the symmetry generators in both theories, and that will be the main theme in the thesis. In particular, the energy of string states should correspond to the conformal dimension of operators in the gauge theory as we will show, and the matching of these quantum numbers in the two theories can provide a verification of the correspondence.

However, calculations are troubled by the fact that it is still unknown how to quantize strings in $AdS_5 \times S^5$. One is then forced to consider a low energy effective description in terms of IIB supergravity, but this approximation is only valid in the limit where $\lambda \gg 1$. On the other hand, perturbative calculations in the gauge theory require $\lambda \ll 1$, and the accessible sectors of the two theories thus seem to be completely incompatible.

If one instead chooses to trust the AdS/CFT correspondence to be true, the difficulties associated with a direct proof become a virtue. The correspondence relates the strongly coupled sector of the planar gauge theory to free string theory in a weakly curved background, where calculations can be performed using classical supergravity. Therefore, it should be possible to gain new insight in the otherwise inaccessible sector of the strongly coupled gauge theory using supergravity. Taking another point of view, the weakly coupled perturbative region of the gauge theory may be able to give clues how to quantize type IIB string theory in a strongly curved gravity background. Or, one can try to incorporate non-planar corrections in the gauge theory to learn about string interactions.[1] The AdS/CFT correspondence is thus a strong/weak duality relating one theory in its weakly coupled perturbative sector with a theory in its strongly coupled non-perturbative sector.

One of the first successful tests of the AdS/CFT correspondence was the work by Mal-

[1]Note that the term string coupling is used in two distinct contexts. The proper string coupling g_s is the world sheet loop counting parameter. In this thesis, we only consider non-interacting strings with $g_s = 0$. However, free strings propagating in a gravity background can be viewed as a two-dimensional interacting field theory with coupling constant $1/\sqrt{\lambda}$.

dacena, Berenstein, and Nastase in 2002 [4]. They considered a limit of the string theory, where the string can be quantized exactly. The limit involves taking $\lambda \gg 1$, so the gauge theory is strongly coupled and it is not possible to perform generic perturbative calculations in λ. The way to circumvent this, is to consider operators that are almost BPS. The conformal dimension of BPS operators are protected from quantum corrections and near-BPS operators only receive small corrections to their conformal dimension. Hence, the authors were able to set up a map between certain string states and operators with matching energies and conformal dimensions.

After the work of BMN, there has been much progress in the calculation of conformal dimensions in $\mathcal{N} = 4$ SYM [5, 6, 7, 8]. Due to operator mixing, one has to diagonalize a matrix of anomalous dimensions and the calculations are highly non-trivial. An important observation was made by Minahan and Zarembo [9] who realized that the problem of finding one-loop planar anomalous dimensions is equivalent to diagonalizing the hamiltonian of an integrable spin chain. Techniques for the diagonalization of spin chain hamiltonians already existed at the time so a new powerful calculational tool was readily at hand. The authors considered the map to a spin chain with $SO(6)$ symmetry, but their results were soon generalized to the full $PSU(2, 2|4)$ super spin chain [10, 11].

Another test of the correspondence has been carried out by Frolov and Tseytlin [12, 13, 14]. The authors investigated a certain family of fast spinning strings, and argued that quantum corrections are suppressed in the limit of large angular momentum and $\lambda/J^2 \ll 1$. Thus, one can compare the classical energy of fast spinning strings with anomalous dimensions obtained perturbatively in the gauge theory.

Recently, Lunin and Maldacena have proposed an extension of the AdS/CFT correspondence to a sector with less symmetry than $AdS_5 \times S^5/\mathcal{N} = 4$ SYM [15]. The extended duality involves a so-called β-deformed version of $\mathcal{N} = 4$ SYM [16], where the supersymmetry has been broken to $\mathcal{N} = 1$, and the authors explain how to obtain the gravity dual of such a field theory. Investigating this deformed version of the correspondence, is clearly an important task since it might reveal the role played by supersymmetry in string/gauge duality.

In this thesis, the energy of spinning strings in Lunin-Maldacena background will be calculated and the result, compared with the anomalous dimension of operators in the marginally deformed $\mathcal{N} = 4$ SYM. The thesis is organized as follows.

Section 2 provides a brief introduction to the AdS/CFT correspondence. We start by giving a short review of the Maldacena conjecture, followed by a discussion of anti-de Sitter space and conformal field theory. It will be shown that the conformal group in four-dimensional Minkowski space is isomorphic to the isometry group of AdS_5 and that the energy of string states in $AdS_5 \times S^5$ corresponds to the conformal dimension of operators in conformal field theory. We also show that the general structure of two-point functions in conformal field theory is uniquely determined by the conformal algebra, and that the conformal dimension of the involved fields can be extracted from these functions. For a detailed discussion and complete list of references, we refer to the reviews [17, 18, 19].

In section 3, the structure of two-point functions is used to construct the one-loop dilatation operator of $\mathcal{N} = 4$ SYM. We start by reviewing the construction of the gauge theory by dimensional reduction, and discuss the symmetries of the theory. Matrix models are introduced, and we show how the complicated combinatorics of two-point functions can be captured in matrix model correlators which simplify significantly in the planar limit. We conclude the section by discussing the β-deformation of $\mathcal{N} = 4$ SYM.

Section 4 is dedicated to the diagonalization of the dilatation operator. The hamiltonian of the Heisenberg spin chain is reviewed and we show that it is equivalent to the $SU(2)$ one-loop dilatation operator in the planar limit. We proceed with the $SU(3)$ spin chain, which requires a little more work. An S-matrix is constructed, and it is shown to satisfy the Yang-Baxter equation implying factorized scattering and integrability. We then apply the nested Bethe ansatz to derive the Bethe equations for the $SU(3)$ spin chain and obtain a rational solution in the thermodynamic limit. Finally, we consider β-deformed spin chains and obtain the anomalous dimension for operators in the deformed theory.

In section 5, the classical energy of strings spinning on the deformed five-sphere is calculated. We start by calculating the classical energy of strings in the undeformed background and then show that the energy of spinning strings in Lunin-Maldacena background can be obtained by a simple substitution of certain winding numbers. The result exactly matches that of the previous section and thus provides evidence that the correspondence indeed remains true when the β-deformation is introduced.

2 AdS/CFT Correspondence

In this section, we start by briefly reviewing the arguments that led Maldacena to his famous conjecture. The basic properties of anti-de Sitter space are then discussed, and it is shown that the boundary of its conformal compactification can be identified with the conformal compactification of Minkowski space. Isometry transformations in anti-de Sitter space act as conformal transformations in Minkowski space, and the basic properties of the conformal group and conformal field theory are reviewed. In particular, we find the general structure of two-point correlation functions in conformal quantum field theories, and this result will be applied in the following section to construct the dilatation operator of $\mathcal{N} = 4$ SYM.

2.1 The Maldacena Conjecture

The Maldacena conjecture states the equivalence of type IIB string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM with gauge group $U(N)$ in four-dimensional Minkowski space. We will now sketch the arguments that led to the conjecture and relate the gauge coupling g_{YM} to the string coupling g_s.

2.1.1 Strings and D-branes

Dp-branes are extended objects with p spatial dimensions that appear in open string theories when one imposes Dirichlet boundary conditions on the strings. The endpoints of the strings are "attached" to the D-brane and can only move tangential to this. In superstring theory, the D-branes are charged and naturally couple to a $(p+1)$-form field potential. For example, in type IIB string theory, the R-R sector gives rise to a totally antisymmetric field $A_{\mu\nu\rho\sigma}$ that couples to D3-branes. On the other hand, the D-branes themselves give rise to a flux of five-form field strength. This is a very natural generalization of electrically charged particles, which is analogous to D0-branes: An electrically charged particle couples to a one-form field potential (which is the usual gauge field A_μ) and gives rise to a flux of two-form field strength (which is just the electric or magnetic fields $F_{\mu\nu}$). Like strings, D-branes can be characterized by a brane tension, which is inversely proportional to the string coupling. They are thus massive objects with a mass given by the product of the brane tension and the volume of the brane.

Consider now type IIB string theory in flat ten-dimensional Minkowski space with a stack of N coincident D3-branes. The theory contains both closed strings and open strings, since the D-branes act as topological defects where open strings can close and vice versa. The massive string states have masses proportional to the string tension, so if we consider the low energy limit, where $E \ll 1/\sqrt{\alpha'}$, only massless states can be excited. In this limit, the open string states are massless excitations on the branes and are described by $\mathcal{N}=4$ SYM with gauge group $U(N)$ on the four-dimensional world volume of the branes, whereas the closed string states are described by type IIB supergravity in the ten-dimensional bulk. In the low energy limit, the massless open and closed string excitations do not interact [17] and we thus have two decoupled theories.

Next, we take a different point of view. Since D-branes are massive charged objects, they give rise to various supergravity fields. In particular, they induce a curvature in spacetime, and the metric of N coincident D3-branes can be calculated in classical supergravity[2] [17]:

$$ds^2 = \Big(1 + \frac{R^4}{u^4}\Big)^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \Big(1 + \frac{R^4}{u^4}\Big)^{\frac{1}{2}}(du^2 + u^2 d\Omega_5^2), \tag{2.1}$$

where

$$R^4 = 4\pi g_s \alpha'^2 N, \tag{2.2}$$

and $u^2 d\Omega_5^2$ is the metric of a five-sphere with radius u. The solution has a horizon, which is at the end of an infinite throat as shown in figure 1. The D3-branes appear as a point in the six spatial dimensions transverse to the world-volume of the branes, and in the six-dimensional transverse space, the D3-branes are encompassed by the five-sphere.

We see that in the limit where $u \gg R$, the geometry becomes asymptotically flat. On the other hand, when $u \ll R$, the radius of curvature of the five-sphere approach R and the metric in the near horizon region becomes

$$ds^2 = \frac{u^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\Big(\frac{du^2}{u^2} + d\Omega_5^2\Big), \tag{2.3}$$

which is exactly the geometry of $AdS_5 \times S^5$ as will be shown in section 2.2.3.

[2]Throughout the thesis we will be using fundamental units where $\hbar = c = 1$.

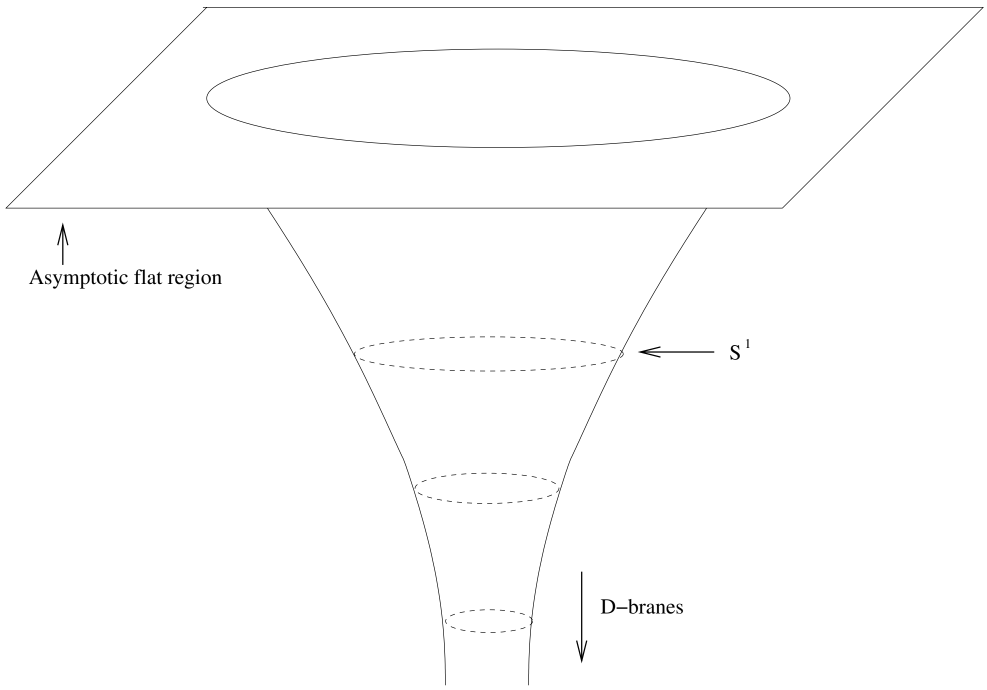

Figure 1: The two-dimensional version of the throat geometry given in (2.1). The surface is a stack of one-spheres with radius u approaching a constant radius R far down the throat. In the ten-dimensional case, the circles are replaced by five-spheres, which encompass the four-dimensional space spanned by t, x_1, x_2 and x_3.

Since G_{tt} depends on u, the energy as measured by an observer far from the branes ($G_{tt} = 1$) is related to the energy measured by an observer at u by

$$E_\infty = \left(1 + \frac{R^4}{u^4}\right)^{-\frac{1}{4}} E_u. \tag{2.4}$$

If one observer is near the horizon ($u \ll R$), we get the relation $E_\infty = \frac{u}{R} E_u$, and the observer far from the branes will observe all energies near the horizon as being small. For an observer far from the branes, there will then be two kinds of low energy excitations: Genuine low energy excitations far from the branes where space is asymptotically flat, and any excitations in the near horizon region of the spacetime. The near horizon theory is IIB string theory in $AdS_5 \times S^5$, and the theory far from the branes is supergravity in flat ten-dimensional spacetime. The two theories decouple in the low energy limit [17].

The low energy limit of IIB string theory in a background of N D3-branes has now been considered from two different angles. In both cases, the theory decouples into two pieces, and one of these is supergravity in flat ten-dimensional spacetime. It is now natural to identify the second theory appearing in the two descriptions, and this is how Maldacena was led to the conjecture: $\mathcal{N} = 4$ SYM in four-dimensional Minkowski space is dual to type IIB string theory in $AdS_5 \times S^5$.

2.1.2 Born-Infeld Electrodynamics

The lagrangian of gauge fields living on D-branes is not given by the usual field strength $F_{\mu\nu}F^{\mu\nu}$, but a generalization that reduces to the well known quadratic term in the small field limit. For simplicity, we will just consider an abelian gauge field in four dimensions. One can then use T-duality to argue that there is a maximal limit to the magnitude of the electric field given by $E_{max} = 1/(2\pi\alpha')$. This fact has the nice implication that a point charge has a finite self-energy. To set up a lagrangian that incorporates a maximal electric field we can let us inspire by the lagrangian of a free relativistic particle

$$L = -m\sqrt{1-v^2}, \tag{2.5}$$

where the requirement that velocities (in fundamental units) cannot exceed 1 is implicit by the positivity of the argument of the square root. Using T-duality, it can be argued that a consistent lagrangian density describing gauge fields on D-branes is given by the Born-Infeld lagrangian [20]

$$\mathcal{L} = -T_3\sqrt{-det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \tag{2.6}$$

where T_3 is the tension of a D3-brane and $\eta_{\mu\nu}$ is the metric tensor of four-dimensional Minkowski space with signature $(-,+,+,+,)$. This lagrangian is Lorentz invariant as can be seen by writing the Lorentz transformed determinant in matrix notation

$$det[(\eta' + F')] = det[L(\eta + F)L^T] = det(\eta + F), \tag{2.7}$$

where L is a Lorentz transformation matrix with $LL^T = I$. If we consider a field strength where only E_x is non-vanishing, the determinant is $-1 + (2\pi\alpha' E_x)^2$, and the Born-Infeld lagrangian (2.6) can be expanded to second order in α' giving[3]

$$\mathcal{L} = -T_3\Big(1 - \frac{(2\pi\alpha' E_x)^2}{2}\Big) = -T_3\Big(1 + \frac{(2\pi\alpha')^2}{4}F_{\mu\nu}F^{\mu\nu}\Big). \tag{2.8}$$

The tension of a Dp-brane is related to the string coupling g_s by [20]

$$T_p = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha' g_s}, \tag{2.9}$$

and insisting that the terms quadratic in the field strength should be given by the term[4] $\frac{-1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu}$, we get an expression for the gauge theory coupling constant in four dimensions in terms of the string coupling:

$$g_{YM}^2 = 4\pi g_s. \tag{2.10}$$

[3]We will be using the Einstein summation convention throughout this thesis. All repeated indices are summed.

[4]The reason we have a 2 instead of the more conventional 4 in the denominator, is that we will be using the normalization $Tr[T_aT_b] = \frac{1}{2}\delta_{ab}$ for the non-abelian $U(N)$ generators. This produces the term $\frac{-1}{4g_{YM}^2}F^a_{\mu\nu}F^{\mu\nu}_a$ in the non-abelian gauge theory.

The exact same result is obtained using non-abelian gauge fields. The only difference is that both the Yang-Mills term and the Born-Infeld lagrangian involve a trace over group indices. Comparing with (2.2), we see that the gauge theory coupling constant is related to the radius of curvature R by

$$\frac{R^4}{\alpha'^2} = g_{YM}^2 N = \lambda. \tag{2.11}$$

In this thesis we will mostly be concerned with the planar limit of the gauge theory and non-interacting strings where

$$N \to \infty, \qquad g_{YM} \to 0, \qquad \lambda = \text{fixed}. \tag{2.12}$$

2.2 Anti-de Sitter Space

One of the remarkable features of the AdS/CFT correspondence is that it provides a holographic description of gravity. In general, this means that the theory of gravity on a given manifold is described by a theory without gravity on the boundary of the manifold. The basic principle is not as abstract as it may sound. Consider for example a massless scalar field φ that occupies some volume V in flat space and is a solution to the Laplace equation $\nabla^2\varphi = 0$. When ϕ is given on the boundary of V, there is a unique extension to the rest of V, as is well known from for example electromagnetism. A holographic description of gravity is of course much more complicated, since it is then a complete (field) theory on the boundary of V that should have a unique extension to V itself. In the AdS/CFT correspondence, the spacetime containing gravity is anti-de Sitter space, but since anti-de Sitter space is not a compact space, we first need to specify exactly what is meant when we speak of its boundary.

In this subsection, we start by defining maximally symmetric spaces and their isometry groups. We then discuss Minkowski space and how its conformal compactification can be related to a sphere. Finally we show how to conformally compactify anti-de Sitter space and associate Minkowski space with its boundary.

2.2.1 Maximally Symmetric Spaces

Symmetry transformations of a manifold are referred to as isometries and can be thought of as coordinate transformations that leave the geometry of the manifold invariant. A precise definition involves the notion of Killing vectors [21], but since the manifolds we consider are quite simple, we will take a more practical point of view. We simply embed a given curved space as a hypersurface in a higher dimensional flat space and define isometries as those coordinate transformations of embedding space that leave the hypersurface invariant.

Let us first consider Euclidian d-dimensional flat space $\mathbb{R}^d$, where the isometries are rotations and translations. There is d coordinate axes and each axis can be rotated into the $d-1$ remaining axes, but then we have counted every rotation twice, so the number of independent rotations is $\frac{1}{2}d(d-1)$. The addition of d translations gives a total of $\frac{1}{2}d(d+1)$

isometries. Any d-dimensional space with this number of isometries is said to be maximally symmetric [21]. An example of a curved maximally symmetric space is the d-dimensional sphere S^d. This can be seen by embedding the sphere in $d+1$-dimensional space and note that there are $\frac{1}{2}d(d+1)$ rotations that leave the sphere invariant.

If we change the signature of the metric, the number of isometries remains the same and Minkowski space is thus another example of a maximally symmetric space. In fact, any d-dimensional space, that can be embedded in a $(d+1)$-dimensional flat space as a hyperboloid fulfilling $\eta_{MN}X^M X^N = R^2$, where η_{MN} is diagonal and has the signature $(++\ldots--\ldots)$ with p pluses and q minuses and $p+q=d+1$, is a maximally symmetric space. The set of isometries on such spaces form a group called the generalized orthogonal group, which is denoted $SO(p,q)$. This group reduces to the ordinary orthogonal group if q is zero and therefore contains this group as a special case. Every element in $SO(p,q)$ can be constructed from the $\frac{1}{2}d(d+1)$ generators $J_{MN} = -J_{NM}$, where $M, N \in \{1, 2, \ldots, d+1\}$. The commutators of these generators are

$$[J_{MN}, J_{RS}] = i(\eta_{NR}J_{MS} + \eta_{MS}J_{NR} - \eta_{MR}J_{NS} - \eta_{NS}J_{MR}), \tag{2.13}$$

where η_{MN} is diagonal with (p,q) signature.

The Ricci scalar R is constant on a maximally symmetric space, and the Riemann tensor is given by [21]

$$R_{\rho\sigma\mu\nu} = \frac{R}{d(d-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}). \tag{2.14}$$

This means that locally, space is fully specified by R and since the magnitude of R just represents an overall scaling, we can characterize a maximally symmetric space by whether R is zero, positive, or negative.

If we stick to Lorentz signature $(-+++\ldots)$, we know that a maximally symmetric space with $R=0$ is simply Minkowski space. The space with positive curvature $(R>0)$ is called de Sitter space and can be embedded in $(d+1)$-dimensional Minkowski space as the hyperboloid given by

$$-X_0^2 + X_1^2 + X_2^2 + \ldots + X_{d+1}^2 = r^2, \tag{2.15}$$

where r is the radius of curvature. This space can be thought of as a Lorentz-sphere, since it is the direct analog of S^d in Euclidian space. Similarly, the maximally symmetric space with negative curvature $(R<0)$ is called Anti-de Sitter space and can be embedded as the hyperboloid given by

$$-X_0^2 + X_1^2 + \ldots + X_d^2 - X_{d+1}^2 = -r^2. \tag{2.16}$$

De Sitter and anti-de Sitter space are empty space solutions to the Einstein equations with a positive and negative cosmological constant, respectively.

2.2.2 Minkowski Space

To analyze the global structure of a manifold, it will be convenient to map the entire manifold onto a diagram using coordinates that have a finite range. For example, the metric of two-dimensional Minkowski space

$$ds^2 = -dt^2 + dx^2, \tag{2.17}$$

where $-\infty < t < \infty$ and $-\infty < x < \infty$, can be written

$$ds^2 = -\frac{1}{\cos^4 t'} dt'^2 + \frac{1}{\cos^4 x'} dx'^2, \tag{2.18}$$

where $t' = \arctan t$ and $x' = \arctan x$, and

$$-\frac{\pi}{2} < t' < \frac{\pi}{2}, \quad -\frac{\pi}{2} < x' < \frac{\pi}{2}. \tag{2.19}$$

This looks promising, since we are now able to draw the entire spacetime as an unbounded square with side length π. However, the causal structure of spacetime is not apparent since light ray trajectories ($ds^2 = 0$) are no longer straight lines at right angles. We are interested in transformations that involve a timelike coordinate T and a spacelike coordinate X, that maintains $dX/dT = \pm 1$. A spacetime diagram with such coordinates is called a conformal diagram (it conserves the form of the lightcones).

In the case of two-dimensional Minkowski space, the trick is to use the lightcone coordinates

$$u = t - x, \tag{2.20}$$

$$v = t + x, \tag{2.21}$$

$$-\infty < u < \infty, \quad -\infty < v < \infty, \tag{2.22}$$

with which the metric becomes

$$ds^2 = -du dv. \tag{2.23}$$

We then obtain a finite range by the change of coordinates

$$U = \arctan u, \tag{2.24}$$

$$V = \arctan v, \tag{2.25}$$

$$-\frac{\pi}{2} < U < \frac{\pi}{2}, \quad -\frac{\pi}{2} < V < \frac{\pi}{2}, \tag{2.26}$$

and the metric becomes

$$ds^2 = \frac{-dU dV}{\cos^2 U \cos^2 V}. \tag{2.27}$$

Reintroducing timelike and spacelike coordinates

$$T = V + U, \tag{2.28}$$
$$X = V - U, \tag{2.29}$$
$$-\pi < X < \pi, \quad |T| < \pi \pm X, \tag{2.30}$$

we get

$$ds^2 = \frac{1}{(\cos X + \cos T)^2}\Big[-dT^2 + dX^2\Big] \equiv \omega^{-2}(T,X)\widetilde{ds^2}, \tag{2.31}$$

where we have defined the rescaled metric $\widetilde{ds^2}$ as the expression in the square brackets above and $\omega(T,X) = \cos X + \cos T$.

A transformation that acts on the metric as a rescaling:

$$ds^2(X,T) \to \omega^2(X,T)ds^2(X,T), \tag{2.32}$$

leaves angles invariant and is called a Weyl transformation. In particular, light ray trajectories are left invariant under such a transformation and therefore, we can capture the causal structure of spacetime just by considering $\widetilde{ds^2}$. Again, Minkowski space described by the coordinates T and X can be depicted as an unbounded square, but now the corners are positioned at $\pm\pi$ at the T and X axes. The boundary of this square corresponds to infinity in the original coordinates and is called conformal infinity. The union of the original spacetime with conformal infinity gives a bounded space called the conformal compactification of spacetime.[5]

A convenient way of picturing the conformal compactification of two-dimensional Minkowski space is to extend the range of T to $-\infty < T < \infty$, and identifying $X = \pi$ with $X = -\pi$. The space is then $\mathbb{R} \times S$ and can be visualized as the surface of a cylinder as shown in figure 2.

The generalization to four-dimensional Minkowski space is straightforward. Using spherical coordinates the metric can be written

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2, \tag{2.33}$$

where $d\Omega_2$ is the metric on a unit two-sphere. One can make the same transformations as above, but now using the radial coordinate r instead of x. Since r has a different range than x ($0 \leq r < \infty$), we also get a different range for the spacelike coordinate R in the end. Performing the transformation yields the metric

$$ds^2 = \frac{1}{4(\cos X \cos T)^2}\Big[-dT^2 + dR^2 + \sin^2 R d\Omega_2^2\Big] \equiv \omega^{-2}(T,R)\widetilde{ds^2}, \tag{2.34}$$

[5]We will only be considering spatial compactification, since we will extend the range of T to cover all of $\mathbb{R}$. When speaking of conformal compactifications in the following, it is understood that we are compactifying the spacelike part of spacetime.

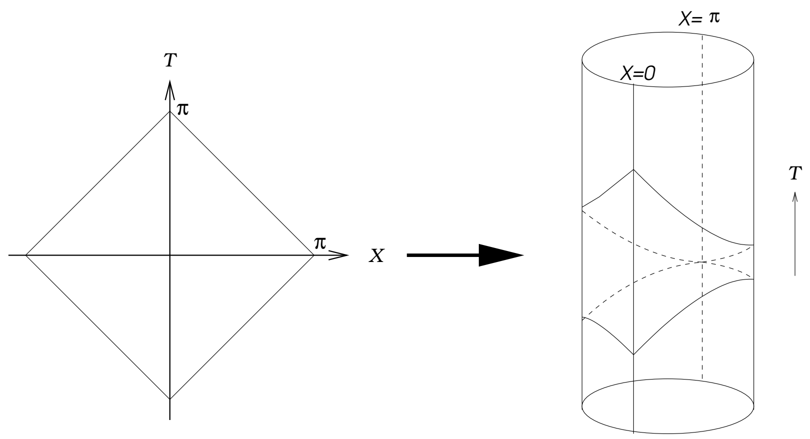

Figure 2: The conformal compactification of two-dimensional Minkowski space can be embedded on the surface of a cylinder. The boundary of the square is conformal infinity. Extending the range of T to $\mathbb{R}$ gives the universal cover of compactified Minkowski space.

where the coordinates have ranges

$$0 \leq R < \pi, \quad |T| + R < \pi. \tag{2.35}$$

The conformal diagram corresponding to these coordinates is now the right half of square in figure 2 with X substituted by R and the boundary $R = 0$ included. The rescaled metric in (2.34) contains the term $dR^2 + \sin^2 R d\Omega_2^2$, which is the metric of a three-sphere. As before, we extend the range of T to all of $\mathbb{R}$ and add the point $R = \pi$ to obtain the conformal compactification of four-dimensional Minkowski space. This space has geometry of $\mathbb{R} \times S^3$ and can be thought of as an infinite timeline where each point represents a spatial three-sphere.[6]

2.2.3 Anti-de Sitter Space

We will now conformally compactify anti-de Sitter space the same way we just did with Minkowski space. To be specific, we consider five-dimensional anti-de Sitter space which can be defined as the hyperboloid

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 = -R^2, \tag{2.36}$$

[6]We note that $0 \leq R \leq \pi$ is exactly enough to cover S^3 whereas we needed $-\pi \leq X < \pi$ to cover S.

embedded in a six-dimensional pseudo-euclidian space. We start by giving the parametrization that results in the metric (2.3):

$$\begin{aligned} X_0 &= \frac{R^2}{2u}\Big(1+\frac{u^2}{R^2}+u^2\frac{\eta_{\mu\nu}x^\mu x^\nu}{R^4}\Big), & X_1 &= \frac{u}{R}x_1, \\ X_2 &= \frac{u}{R}x_2, & X_3 &= \frac{u}{R}x_3, \\ X_4 &= \frac{R^2}{2u}\Big(1-\frac{u^2}{R^2}+u^2\frac{\eta_{\mu\nu}x^\mu x^\nu}{R^4}\Big), & X_5 &= \frac{u}{R}t, \end{aligned} \tag{2.37}$$

with $u > 0,\ x_\mu \in \mathbb{R}$ and $\eta_{\mu\nu}x^\mu x^\nu = -t^2 + x_1^2 + x_2^2 + x_3^2$. This is seen to solve (2.36), and the induced metric becomes

$$ds^2 = \frac{u^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + R^2\frac{du^2}{u^2}. \tag{2.38}$$

In this form of the metric, the Poincaré symmetry of x^μ is manifest. The metric is also invariant under the $SO(1,1)$[7] transformation $(u, x^\mu) \to (k^{-1}u, kx^\mu)$, and this isometry is defined as the dilatation of the conformal group in section 2.3.

The parametrization we will use to conformally compactify AdS_5 is given by

$$X_0 = R\cosh\rho\cos t, \tag{2.39}$$
$$X_1 = R\sinh\rho\cos\psi\sin\vartheta_1, \tag{2.40}$$
$$X_2 = R\sinh\rho\cos\psi\cos\vartheta_1, \tag{2.41}$$
$$X_3 = R\sinh\rho\sin\psi\sin\vartheta_2, \tag{2.42}$$
$$X_4 = R\sinh\rho\sin\psi\cos\vartheta_2, \tag{2.43}$$
$$X_5 = R\cosh\rho\sin t, \tag{2.44}$$

and is also seen to solve (2.36). A two-dimensional version of anti-de Sitter space is drawn in figure 3. The metric induced from this parametrization is

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2), \tag{2.45}$$

where $d\Omega_3$ is the metric of a unit three-sphere. From the metric, it is clear that t is a timelike coordinate as was already indicated by using the letter t, and we note that it has a period of 2π. This peculiar periodic behavior stems from the way we defined the space as embedded in a higher dimensional space. We can define five-dimensional anti-de Sitter space by the metric (2.45), and we are then free to consider the "unwrapped" anti-de Sitter space and take the range of t to all of $\mathbb{R}$. This is the universal cover of AdS_5, which we will have in mind when referring to anti-de Sitter space.

[7] The group $SO(1,1)$ can be defined as the set of transformations that leave the bilinear form $X^2 - Y^2$ invariant. Defining u and v by $2X = (u+v)$ and $2Y = (u-v)$, transforms the bilinear form into uv, which is invariant under $(u,v) \to (k^{-1}u, kv)$.

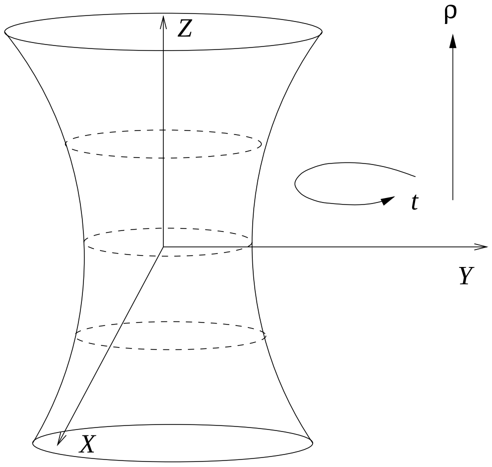

Figure 3: Two-dimensional anti-de Sitter space (AdS_2) embedded in three-dimensional Minkowski space. The surface is given by $X^2+Y^2-Z^2=R^2$ and can be parameterized by $X = R\cosh\rho\cos t$, $Y = R\cosh\rho\sin t$, $Z = R\sinh\rho$, which gives rise to timelike closed curves in the XY-plane. Conformal infinity is disconnected, but this is a special feature of AdS_2, where we have to take $\rho \in \mathbb{R}$ to cover the hyperboloid. In higher dimensional anti-de Sitter space we take $\rho \geq 0$.

To obtain the conformal compactification of AdS_5, we introduce a new coordinate χ, defined by $\tan\chi = \sinh\rho$ $(0 \leq \chi < \pi/2)$ with which the metric can be written

$$ds^2 = \frac{R^2}{\cos^2\chi}(-dt^2 + d\chi^2 + \sin^2\chi d\Omega_3^2). \tag{2.46}$$

The conformally rescaled metric is that of $\mathbb{R} \times S^4$, but in contrast to the conformally compactified Minkowski space, this space is not covered by the range of the coordinates. Only half of S^4 is covered since $0 \leq \chi < \pi/2$ rather than $0 \leq \chi < \pi$. We add the boundary of the S^4 hemisphere ($\chi = \pi/2$) to get the conformally compactified anti-de Sitter space. This boundary is exactly S^3 and conformal infinity of AdS_5 thus have the geometry of $\mathbb{R} \times S^3$. In other words, the boundary of the conformally compactified AdS_5 is identical to the conformal compactification of four-dimensional Minkowski space. This identification is shown in figure 4 for AdS_3 and two-dimensional Minkowski space.

Since the boundary of the conformally compactified AdS_5 is the four-dimensional conformal compactification of Minkowski space, it is not immediately clear how the isometry group $SO(2,4)$ acts on this hypersurface. One can work this out by parameterizing points on the boundary by "AdS-like" coordinates and then consider an infinitesimal $SO(2,4)$ transformation on these [19]. The result is that $SO(2,4)$ generates conformal transformations in Minkowski space. Since AdS_5 is invariant under $SO(2,4)$-transformations, it is expected that the theory living on Minkowski space should be invariant to conformal transformations. In the next section, we will define conformal transformations, discuss exactly what it means for a theory to be conformally invariant, and show explicitly that

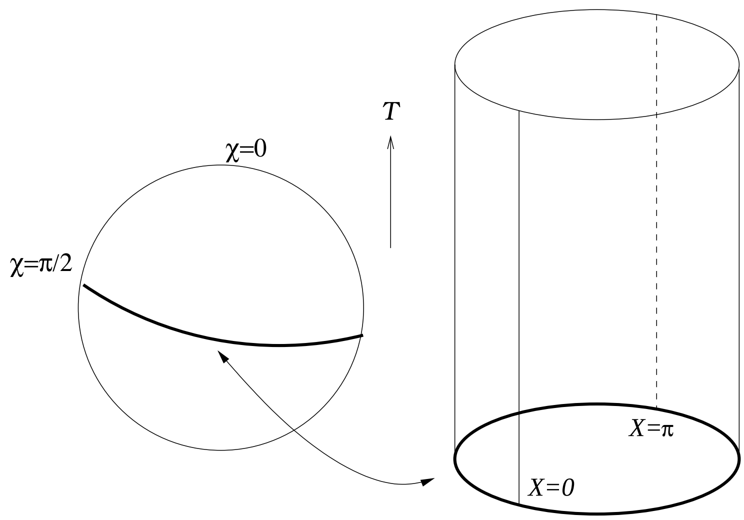

Figure 4: The spatial part of the conformal compactification of AdS_3 can be pictured as the upper half of a two-sphere, whereas two-dimensional Minkowski space can be depicted as the surface of a cylinder. At a given time, the conformal compactification of Minkowski space is a circle and can be identified with conformal infinity of AdS_3 which is the circle at $\chi = \pi/2$. In the conformal compactification of AdS_5 and four-dimensional Minkowski space, we just replace the circles that constitute the cylinder with three-spheres.

the conformal group is isomorphic to $SO(2, 4)$.

2.3 Conformal Field Theory

In relativistic field theory, Poincaré invariance obviously plays a prominent role, reflecting that spacetime is homogeneous and that different Lorentz observers should observe the same physical laws. The set of Poincaré transformations form a group, and a natural question one could ask is whether the Poincaré group can be generalized to a larger symmetry group of a theory. The Coleman-Mandula theorem states that there are no such bosonic generalizations of the Poincaré group consistent with the existence of an S-matrix. However, if one imposes scale invariance on a theory, it is no longer possible to define an S-matrix in the usual sense, since the notion of asymptotic states becomes obscure. Therefore, scale invariance is not in conflict with the Coleman-Mandula theorem and as we will see, scale transformations indeed enlarge the Poincaré group in a natural way.

In this subsection, the generators of the conformal coordinate transformations are derived and the commutation relations defining the conformal algebra are given. It is then shown that representations of the conformal algebra can be labeled by a scaling dimension, and the action of the conformal generators on such fields is derived. In the end, we show that the form of correlation functions in conformal quantum field theory are severely restricted by the conformal invariance of the theory.

2.3.1 Conformal Transformations

As its name implies, a conformal transformation is a coordinate transformation that leaves angles invariant and thus conserves the local geometry of spacetime. It can be defined as a transformation which acts on the metric as a scale transformation [22]:

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{2.47}$$

This set of transformations includes Poincaré transformations as a special case, since these simply leave the metric invariant. Under a general coordinate transformation, the metric transforms as according to

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x), \tag{2.48}$$

and by comparing (2.47) and (2.48), we can derive the general form of conformal coordinate transformations. We consider the infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$ corresponding to an infinitesimal conformal transformation and require that the metric transforms as $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\mu\nu}(x) + \omega(x) g_{\mu\nu}(x)$. Using (2.48), we get to first order in $\epsilon(x)$

$$g_{\mu\nu}(x) - \partial_\mu \epsilon_\nu(x) - \partial_\nu \epsilon_\mu(x) = g_{\mu\nu}(x) + \omega(x) g_{\mu\nu}(x). \tag{2.49}$$

Taking the trace of this equation, we get that $\omega(x) = -\frac{2}{d}\partial_\mu \epsilon^\mu(x)$, where d is the number of spacetime dimensions. Inserting $\omega(x)$ into (2.49), we obtain the following equation for $\epsilon(x)$

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}\partial_\sigma \epsilon^\sigma(x) g_{\mu\nu}(x), \tag{2.50}$$

which has the general solution

$$\epsilon^\mu(x) = \alpha^\mu + \omega^\mu_\nu x^\nu + \sigma x^\mu + \beta^\mu x^2 - 2x^\mu \beta_\nu x^\nu, \tag{2.51}$$

for $d > 2$ (see appendix A). Here, $\omega_{\mu\nu}$ is antisymmetric, and α_μ and β_μ are arbitrary vectors. There is then four types of conformal coordinate transformations corresponding to the four infinitesimal parameters above:

Translations:	$x^\mu \to x^\mu + \alpha^\mu$	(2.52)
Lorentz transformations:	$x^\mu \to x^\mu + \omega^\mu_\nu x^\nu$	(2.53)
Dilatations:	$x^\mu \to x^\mu + \sigma x^\mu$	(2.54)
Special conformal transformations:	$x^\mu \to x^\mu + \beta^\mu x^2 - 2x^\mu \beta_\nu x^\nu.$	(2.55)

Whereas the first two are the infinitesimal form of the well known Poincaré transformations corresponding to $\Omega = 1$ in (2.47), the dilatation and special conformal transformation are genuine rescalings of the metric. The transformations can be exponentiated to obtain the following finite forms:

$$\text{Translations:} \qquad x^\mu \to x^\mu + a^\mu \tag{2.56}$$

$$\text{Lorentz transformations:} \qquad x^\mu \to L^\mu_\nu x^\nu \tag{2.57}$$

$$\text{Dilatations:} \qquad x^\mu \to \lambda x^\mu \tag{2.58}$$

$$\text{Special conformal transformations:} \qquad x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b_\nu x^\nu + b^2 x^2}. \tag{2.59}$$

The dilatation is simply a rescaling of spacetime itself, and one could easily have guessed this transformation from (2.47) and (2.48). The special conformal transformation corresponds to a shift of b^μ preceded and followed by the inversion $x^\mu \to \frac{x^\mu}{x^2}$ and will take infinity to the finite point $-b^\mu/b^2$.

In addition to the usual $\frac{1}{2}d(d+1)$ parameters associated with Poincaré transformations, there is also $d+1$ parameters coming from the dilatation and special conformal transformations giving a total of $\frac{1}{2}(d+1)(d+2)$ parameters. We should remember though, that (2.51) is only valid for $d > 2$. If $d = 1$, the notion of conformal symmetry does not make much sense since the metric is a number and all general coordinate transformations have the form (2.47). For $d = 2$, there is an infinite number of parameters associated with the transformation.

2.3.2 The Conformal Algebra

Let us first consider a field, which is invariant under conformal transformations. Such a field can be thought of as a conformal scalar and one simply has

$$\varphi'(x') = \varphi(x). \tag{2.60}$$

This kind of field is not very interesting for our purpose, but it can be used to derive the commutation relations for the conformal generators. We define the conformal generator G_a associated with an infinitesimal conformal transformation according to

$$\varphi'(x) = \varphi(x) - i\delta_a G_a \varphi(x), \tag{2.61}$$

where δ_a denotes the $\frac{(d+1)(d+2)}{2}$ parameters characterizing the transformation. With the infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$, equation (2.60) becomes

$$\varphi'(x') = \varphi(x' - \epsilon) = \varphi(x') - \epsilon^\mu \partial_\mu \varphi(x'), \tag{2.62}$$

to first order in $\epsilon(x)$. Comparing with (2.61) and using (2.51) we see that the generators corresponding to the four types of conformal generators are

$$\text{Translations:} \qquad \tilde{P}_\mu \varphi(x) = -i\partial_\mu \varphi(x) \tag{2.63}$$

$$\text{Lorentz transformations:} \qquad \tilde{M}_{\mu\nu}\varphi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi(x) \tag{2.64}$$

$$\text{Dilatations:} \qquad \tilde{D}\varphi(x) = -ix^\mu \partial_\mu \varphi(x) \tag{2.65}$$

$$\text{Special conformal transformations:} \qquad \tilde{K}_\mu \varphi(x) = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)\varphi(x) \tag{2.66}$$

and with these, we can derive the conformal algebra

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \\
[K_\mu, M_{\nu\rho}] &= i(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu) \\
[P_\mu, M_{\nu\rho}] &= i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) \\
[D, M_{\mu\nu}] &= [K_\mu, K_\nu] = [P_\mu, P_\nu] = 0.
\end{aligned} \tag{2.67}$$

We will now regard these commutators as fundamental abstract objects characterizing conformal transformations in general and not just the simple coordinate transformations considered above. The differential generators (2.63)-(2.66) furnish a representation of the conformal algebra and have been equipped with tildes to emphasize that this is just one of many possible representations.

In four dimensions, the conformal algebra is isomorphic to the algebra of $SO(4, 2)$ given by (2.13), as can be seen by setting

$$J_{\mu\nu} = M_{\mu\nu}, \qquad J_{54} = D, \qquad J_{\mu 4} = \frac{1}{2}(P_\mu - K_\mu), \qquad J_{\mu 5} = \frac{1}{2}(P_\mu + K_\mu), \tag{2.68}$$

where J_{MN} has the associated metric $\eta_{MN} = diag(-++++-)$, and $M, N \in \{0, 1, 2, 3, 4, 5\}$.

2.3.3 Classical Conformal Field Theory

We will be interested in fields with a definite conformal dimension Δ, that transforms according to

$$\Phi(x) \to \Phi'(x') = \lambda^{-\Delta}\Phi(\lambda x) \tag{2.69}$$

under dilatations. The infinitesimal form of such a transformation can be written (with $\lambda = 1 + \sigma$)

$$\Phi(x) \to (1 - i\sigma D)\Phi(x') = (1 - \sigma\Delta)\Phi((1 + \sigma)x) \tag{2.70}$$

and the fields are thus eigenstates of the dilatation generator with eigenvalue $-i\Delta$. The conformal dimension is simply the mass dimension of classical fields, but in quantum field theory, the conformal dimension of fields receives corrections called the anomalous dimension, when the theory is renormalized. We note that the dilatation generator commutes with the Lorentz generators, and we should thus be able to assign a conformal dimension to fields that carry a representation of the Lorentz algebra. We will make this dependence explicit by writing Φ_Δ for fields carrying a definite conformal dimension. However, this means that we cannot in general characterize such fields by eigenvalues of the hamiltonian P_0 or the mass operator $M^2 = -P_\mu P^\mu$ which is not a Casimir of the conformal group.

To derive the general transformation properties of conformal fields, we first restrict ourselves to the subalgebra obtained when spacetime translations are excluded from the conformal algebra. We consider the action of these generators on fields at the origin. The Lorentz generators for example are given by its spinor representation

$$M_{\mu\nu}\Phi_\Delta(0) = \Sigma_{\mu\nu}\Phi_\Delta(0), \tag{2.71}$$

where $\Sigma_{\mu\nu}$ is a matrix satisfying the Lorentz algebra and $\Phi_\Delta(0)$ is a multicomponent field. Similarly, we define the action of dilatations and special conformal transformations on fields at the origin by

$$D\Phi_\Delta(0) = D_0\Phi_\Delta(0), \tag{2.72}$$
$$K_\mu\Phi_\Delta(0) = \kappa_\mu\Phi_\Delta(0). \tag{2.73}$$

The subalgebra of generators acting on fields at $x = 0$ is then

$$[D_0, \Sigma_{\mu\nu}] = 0, \tag{2.74}$$
$$[D_0, \kappa_\mu] = -i\kappa_\mu, \tag{2.75}$$
$$[\kappa_\mu, \Sigma_{\nu\rho}] = i(\eta_{\mu\nu}\kappa_\rho - \eta_{\mu\rho}\kappa_\nu), \tag{2.76}$$
$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = i(\eta_{\nu\rho}\Sigma_{\mu\sigma} + \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho}), \tag{2.77}$$
$$[\kappa_\mu, \kappa_\nu] = 0. \tag{2.78}$$

The generators at finite points can now be obtained by translating with $e^{iP_\mu x^\mu}$. Using the Baker-Hausdorff formula and the commutators involving P_μ we get

$$\begin{aligned}
e^{iP_\mu x^\mu}\Sigma_{\mu\nu}e^{-iP_\mu x^\mu}\Phi_\Delta(x) &= \big(\Sigma_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu\big)\Phi_\Delta(x), \\
e^{iP_\mu x^\mu}\kappa_\mu e^{-iP_\mu x^\mu}\Phi_\Delta(x) &= \big(\kappa_\mu + 2x_\mu D_0 - x^\nu\Sigma_{\mu\nu} + 2x_\mu x^\nu P_\nu - x^2 P_\mu\big)\Phi_\Delta(x), \\
e^{iP_\mu x^\mu}D_0 e^{-iP_\mu x^\mu}\Phi_\Delta(x) &= \big(D_0 + x^\mu P_\mu\big)\Phi_\Delta(x).
\end{aligned} \tag{2.79}$$

If we require that Φ_Δ carries an irreducible representation of the Lorentz group, Schurs lemma and (2.74) imply that D_0 is simply a number times the unit matrix. This number is just $-i\Delta$ as noted in (2.70). Using (2.75), we deduce that $\kappa_\mu = 0$ and finally get the action of the generators

$$D\Phi_\Delta(x) = -i\big(\Delta + x^\mu\partial_\mu\big)\Phi_\Delta(x), \tag{2.80}$$
$$P_\mu\Phi_\Delta(x) = -i\partial_\mu\Phi_\Delta(x), \tag{2.81}$$
$$K_\mu\Phi_\Delta(x) = -i\big(2\Delta x_\mu + 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\big)\Phi_\Delta(x) - x^\nu\Sigma_{\mu\nu}\Phi_\Delta(x), \tag{2.82}$$
$$M_{\mu\nu}\Phi_\Delta(x) = -i\big(x_\nu\partial_\mu - x_\mu\partial_\nu\big)\Phi_\Delta(x) + \Sigma_{\mu\nu}\Phi_\Delta(x). \tag{2.83}$$

2.3.4 Conformal Invariance in Quantum Field Theory

In quantum field theory, we replace the classical fields considered above with operators (which we will still call fields). The objects of interest will be correlation functions of fields defined by

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\rangle = \frac{1}{Z}\int \mathcal{D}(\phi_i)\mathcal{O}(x_1)\mathcal{O}(x_2)\ldots e^{iS}, \tag{2.84}$$

where $\mathcal{D}(\phi_i)$ is the functional integration measure of all fields appearing in the action S. The operators $\mathcal{O}(x_i)$ are not necessarily one of the fundamental fields appearing in the functional integral measure, but can represent any composite of these. We will consider correlation functions involving fields carrying a definite conformal, which we denote by $\mathcal{O}_\Delta$. Such correlation functions are highly constrained by the conformal symmetry as we now show.

In quantum field theory, symmetry transformations of fields are given by the commutators of the symmetry generators and the fields. Referring to (2.80)-(2.83), we get the following relations

$$[D, \mathcal{O}_\Delta(x)] = -i\big(\Delta + x^\mu\partial_\mu\big)\mathcal{O}_\Delta(x) \tag{2.85}$$
$$[P_\mu, \mathcal{O}_\Delta(x)] = -i\partial_\mu\mathcal{O}_\Delta(x) \tag{2.86}$$
$$[K_\mu, \mathcal{O}_\Delta(x)] = -i\big(2\Delta x_\mu + 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\big)\mathcal{O}_\Delta(x) - x^\nu\Sigma_{\mu\nu}\mathcal{O}_\Delta(x) \tag{2.87}$$
$$[M_{\mu\nu}, \mathcal{O}_\Delta(x)] = -i\big(x_\nu\partial_\mu - x_\mu\partial_\nu\big)\mathcal{O}_\Delta(x) + \Sigma_{\mu\nu}\mathcal{O}_\Delta(x). \tag{2.88}$$

We define the vacuum to be annihilated by all the symmetry generators. If G is any conformal generator, we then get for a two-point correlation function

$$\begin{aligned} 0 =& \langle 0|G\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)|0\rangle \\ =& \langle 0|\big[G,\mathcal{O}_{\Delta_1}(x_1)\big]\mathcal{O}_{\Delta_2}(x_2)|0\rangle + \langle 0|\mathcal{O}_{\Delta_1}(x_1)G\mathcal{O}_{\Delta_2}(x_2)|0\rangle \\ =& \langle 0|\big[G,\mathcal{O}_{\Delta_1}(x_1)\big]\mathcal{O}_{\Delta_2}(x_2)|0\rangle + \langle 0|\mathcal{O}_{\Delta_1}(x_1)\big[G,\mathcal{O}_{\Delta_2}(x_2)\big]|0\rangle \end{aligned} \tag{2.89}$$

We will use this relation together with (2.85)-(2.87) to determine the form of two-point functions involving conformal fields. We will only be concerned with spinless fields where $\Sigma_{\mu\nu} = 0$, since these are the ones we will be working with later. Let us denote the two-point function by $f_{12}(x_1, x_2)$ and consider $G = P_\mu$. Equations (2.89) and (2.86) then give

$$\Big(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu}\Big) f_{12}(x_1, x_2) = 0. \tag{2.90}$$

If we define the two coordinates $y = x_1 + x_2$, $z = x_1 - x_2$ and consider f_{12} as a function of these new independent coordinates, the above equation says that $f_{12}(x_1, x_2)$ only depends on the difference z and we write $f_{12}(x_1 - x_2) = f_{12}(z)$. Without loosing generality, we can shift one of the coordinates to zero and restrict our attention to the correlation functions $\langle \mathcal{O}_{\Delta_1}(z)\mathcal{O}_{\Delta_2}(0)\rangle = f_{12}(z)$. We now set $G = D$ and get from the above

$$\Big(\Delta_1 + \Delta_2 + z^\mu\frac{\partial}{\partial z^\mu}\Big) f_{12}(z) = 0, \tag{2.91}$$

which has the general solution

$$f_{12}(z) = C_{12}|z|^{-\Delta_1-\Delta_2}. \tag{2.92}$$

Finally, we set $G = K_\mu$ and use the above result to obtain

$$\Big(2\Delta_1 z_\mu + 2z_\mu z^\nu \partial_\nu - z^2\partial_\mu\Big)|z|^{-\Delta_1-\Delta_2} = 0, \tag{2.93}$$

which is only satisfied if $\Delta_1 = \Delta_2$. Two-point functions are then non-vanishing only if the two fields have the same conformal dimension. In that case they are given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}. \tag{2.94}$$

The constant C_{12} depends of the normalization of the fields and has no physical significance. A similar procedure can be used to determine the structure of three-point functions, but they will not be important in this thesis and we will not give the derivation here.

Equation (2.94) is the most important result in this subsection. In the following two sections we will concentrate on the explicit calculation of conformal dimensions of a certain class of operators.

When calculating propagators in quantum field theory, it is often convenient to work in euclidian space instead of Minkowski space. This is obtained by substituting Minkowski time $t_M = x^0$ with the imaginary euclidian time $-ix^4$. The metric is then $\eta_{\mu\nu} = diag(++++)$, where $\mu,\nu \in \{1,2,3,4\}$ and the euclidian version of the conformal algebra is isomorphic to $SO(1,5)$.

The AdS_5 energy or conformal hamiltonian is naturally identified with J_{05} (see (2.39)-(2.44)). When going to euclidian space this generator is exchanged with the dilatation generator:

$$J_{05} \leftrightarrow J_{45}. \tag{2.95}$$

Hence, we expect to match the energy of states in AdS_5 with the conformal dimension of operators.

3 $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

The conformal field theory appearing in the AdS/CFT correspondence is a supersymmetric Yang-Mills theory in four-dimensional Minkowski space, referred to as $\mathcal{N} = 4$ SYM for short. The theory contains the maximal amount of supersymmetry ($\mathcal{N} = 4$) allowed in a theory without gravity and possesses an internal $SU(4)$ R-symmetry.

Our main objective on the gauge theory side of the AdS/CFT correspondence will be to calculate the conformal dimension of operators as a function of the R-charges carried by the operators. We assume that the conformal dimension can be written as a perturbation series in the gauge theory coupling g_{YM}^2, and our goal is to calculate the one-loop correction (the anomalous dimension) in the planar limit. The most important result of this section will be the structure of the one-loop planar dilatation operator, which has anomalous dimensions as eigenvalues.

We start by deriving the action of $\mathcal{N} = 4$ SYM, and comment on the bosonic symmetries of the theory. In particular, the R-symmetries will play an important role, and we define three complex scalar fields with phases conjugate to the three Cartan generators of the R-symmetry algebra. We then calculate two-point correlation functions involving operators that are a product of three complex scalar fields to one-loop, and show how these can be written in terms of matrix model correlators. The fact that two-point correlation functions in $\mathcal{N} = 4$ SYM should coincide with the form of correlation functions in any conformal field theory (given in the last section) is used to define the one-loop dilatation operator. It is then shown that the action of the dilatation operator simplifies considerably in the planar limit and that the anomalous dimension becomes proportional to the 't Hooft coupling $\lambda = N g_{YM}^2$. Finally, we discuss marginal deformations of $\mathcal{N} = 4$ SYM and how the planar dilatation operator is affected by the simplest of such deformations.

3.1 Action and Symmetries

The Lagrangian of $\mathcal{N} = 4$ SYM is uniquely determined by requiring $\mathcal{N} = 4$ supersymmetry and that the theory is renormalizable. It can be constructed from scratch [23], but in the context of string theory and D-branes, it is more natural to derive it by dimensional reduction from $d = 10$ to $d = 4$.

Dp-branes are hyperplanes with p spatial dimensions that arise in theories of open strings when one imposes Dirichlet boundary conditions on the strings. The endpoints of an open string are "attached" to the Dp-brane and can only move tangential to this. The massless spectrum of open bosonic strings gives rise to a Maxwell field living on the $(p+1)$-dimensional world-volume of the Dp-brane and $(d-p-1)$ massless fields transforming as Lorentz scalars on the world-volume of the brane [20]. If we consider a stack of N D-branes on top of each other, there is N^2 possible ways for a given string to have its endpoints attached to a brane and a $U(N)$ gluon field is living on the world-volume of the branes instead of a $U(1)$ Maxwell field. In superstring theory, there will also be massless fermions that interact with the gauge bosons, and the low energy theory on the world-volume of the branes is a supersymmetric Yang-Mills theory.

All the fields in the theory carry the adjoint representation of the gauge group $U(N)$ and are thus given by hermitian $N \times N$ matrices. It is often convenient to expand the fields in terms of the N^2 generators T^a, with which a generic hermitian matrix field can be written

$$\Phi_{\alpha\beta}(x) = \Phi^a(x) T^a_{\alpha\beta}, \tag{3.1}$$

where $a \in \{0, 1, \ldots, N^2-1\}$, and T^0 is the diagonal $U(1)$ generator. We use the conventions of [24], which give the following relations for the generators

$$[T^a, T^b] = if^{abc}T^c, \qquad Tr(T^aT^b) = \frac{1}{2}\delta^{ab}, \qquad T^a_{\alpha\beta}T^a_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{3.2}$$

where f^{abc} are the structure constants of $U(N)$.

It is particularly interesting to consider a stack of D3-branes, since their world volume is four-dimensional Minkowski space. The supersymmetric Yang-Mills theory should contain a gluon field and six scalar fields giving a total of eight on-shell bosonic degrees of freedom. This should also be the number of on-shell fermionic degrees of freedom and can be matched by four Majorana spinors or four Weyl spinors. To find the lagrangian of this theory we consider a stack of D9-branes. Their world volume is simply ten-dimensional spacetime itself and the massless string spectrum does not contain any scalars. The ten-dimensional supersymmetric gauge theory then has an euclidian action given by

$$S_{YM} = \frac{1}{g_{YM}^2}\int d^{10}x\Big[\frac{1}{4}F^a_{MN}F^a_{MN} + \frac{1}{2}\bar{\chi}^a\Gamma_M(D_M\chi)^a\Big], \tag{3.3}$$

with

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + f^{abc}A^b_M A^c_N, \tag{3.4}$$

$$(D_M\chi)^a = \partial_M\chi^a + f^{abc}A^b_M\chi^c, \tag{3.5}$$

and gauge group $U(N)$. This action is invariant under a certain supersymmetry transformation provided we take χ to be a Majorana-Weyl spinor [25]. The gauge theory in four dimensions can be found by dimensional reduction of this theory. We now let $\mu, \nu \in \{0, 1, 2, 3\}$ and $i, j \in \{4, 5, 6, 7, 8, 9\}$ and get for the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu, \tag{3.6}$$

$$F^a_{\mu i} = \partial_\mu A^a_i + f^{abc}A^b_\mu A^c_i = (D_\mu A_i)^a, \tag{3.7}$$

$$F^a_{ij} = f^{abc}A^b_i A^c_j. \tag{3.8}$$

We rename the scalar fields $A_i = \phi_i$ and take $i, j \in \{1, 2, 3, 4, 5, 6\}$. The first term in (3.3) can then be written

$$\frac{1}{4}F^a_{MN}F^a_{MN} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D_\mu\phi_i)^a(D_\mu\phi_i)^a + \frac{1}{4}f^{abc}f^{ade}\phi^b_i\phi^c_j\phi^d_i\phi^e_j. \tag{3.9}$$

With this notation, the spinor term becomes

$$\frac{1}{2}\bar{\chi}^a\Gamma_M(D_M\chi)^a = \frac{1}{2}\bar{\chi}^a\Gamma_\mu(D_\mu\chi)^a + \frac{1}{2}f^{abc}\bar{\chi}^a\Gamma_i\phi^b_i\chi^c, \tag{3.10}$$

and we get the action in four euclidian dimensions

$$S = \frac{1}{g_{YM}^2}\int d^4x\Big\{\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D_\mu\phi_i)^a(D_\mu\phi_i)^a + \frac{1}{4}f^{abc}f^{ade}\phi^b_i\phi^c_j\phi^d_i\phi^e_j + \frac{1}{2}\bar{\chi}^a\Gamma_\mu(D_\mu\chi)^a + \frac{1}{2}f^{abc}\bar{\chi}^a\Gamma_i\phi^b_i\chi^c\Big\}. \tag{3.11}$$

We could also make a dimensional reduction of the Majorana-Weyl spinor χ by writing the 16-dimensional matrices Γ_M in terms of four-dimensional gamma-matrices [25]. This would result in an $SU(4)$ quartet of Weyl spinors, but we prefer to keep the action in the present form. The six scalars carry a sextet of $SU(4)$, but since the Lie algebra of $SU(4)$ is isomorphic to the Lie algebra of $SO(6)$, we can think of this as the vector representation of $SO(6)$. The internal $SU(4)$ symmetry in the action is called R-symmetry.

3.1.1 Conformal Invariance

The action (3.11) can be shown to have classical conformal invariance by using the transformations derived in the last section. Quite obviously, it is invariant under Poincaré transformations so one just needs to demonstrate the invariance under dilatations and special conformal transformations. The fields transform according to

$$D\Phi_\Delta(x) = -i\big(\Delta + x^\mu\partial_\mu\big)\Phi_\Delta(x), \tag{3.12}$$
$$K_\mu\Phi_\Delta(x) = -i\big(2\Delta x_\mu + 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\big)\Phi_\Delta(x) - x^\nu\Sigma_{\mu\nu}\Phi_\Delta(x), \tag{3.13}$$

where Δ is the mass dimension of the fields ($\Delta_F = 2$, $\Delta_\phi = \Delta_A = 1$ and $\Delta_\chi = 3/2$). As an example, we can take field strength term. Under an infinitesimal dilatation, the term will transform like

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} \rightarrow (1 - i\sigma D)F_{\mu\nu}F^{\mu\nu} &= F_{\mu\nu}F^{\mu\nu} - 2\sigma F^{\mu\nu}x^\rho\partial_\rho F_{\mu\nu} - 4\sigma F_{\mu\nu}F^{\mu\nu} \\ &= F_{\mu\nu}F^{\mu\nu} - \sigma\partial_\rho(x^\rho F_{\mu\nu}F^{\mu\nu}). \end{aligned}$$

A total derivative like the second term in the last line can be neglected if we assume the fields vanish at infinity and the transformation thus leaves the action invariant. By the same procedure, we get for an infinitesimal special conformal transformation

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} \rightarrow (1 - i\epsilon^\rho K_\rho)F_{\mu\nu}F^{\mu\nu} &= F_{\mu\nu}F^{\mu\nu} - \epsilon^\rho F_{\mu\nu}(8x_\rho + 4x_\rho x^\sigma\partial_\sigma - 2x^2\partial_\rho)F^{\mu\nu} \\ &= F_{\mu\nu}F^{\mu\nu} - 2\epsilon^\rho\partial_\sigma(x^\sigma x_\rho F_{\mu\nu}F^{\mu\nu}) + \epsilon^\rho\partial_\rho(F_{\mu\nu}F^{\mu\nu}x^2), \end{aligned}$$

and again, we neglect the total derivatives and conclude that this term is invariant under conformal transformations. The other terms in the action can be shown to be conformally invariant by the same procedure, but we will not go through all of them.

Usually, conformal invariance is broken when field theories are quantized, since the coupling constant becomes scale dependent under renormalization. The 1-loop β-function for a gauge theory with N_s real scalars and N_f Dirac fermions is given by [25]

$$\beta(g_{YM}) = \frac{g_{YM}^3}{(4\pi)^2}\Big[-\frac{11}{3}c_g + \frac{1}{6}N_s c_s + \frac{4}{3}N_f c_f\Big], \tag{3.14}$$

where the c's are determined by the representation of the various fields by $Tr(T^aT^b) = c\delta^{ab}$. In $\mathcal{N} = 4$ SYM, the complete β-function is given by the 1-loop contribution [23], and since all the fields are in the same representation and with two Dirac spinors and six scalars, the β-function vanishes. Therefore, the coupling constant of $\mathcal{N} = 4$ SYM does not run and conformal invariance is maintained in the quantum theory.

3.1.2 R-Charges

The scalar fields are in the vector representation of $SO(6)$ and we should thus be able to label fields involving scalars by quantum numbers associated with this group. The Cartan subalgebra of $SO(6)$ is three-dimensional, and an obvious choice of basis is the three commuting charges $J_1 \equiv J_{12}$, $J_2 \equiv J_{34}$ and $J_3 \equiv J_{56}$ belonging to the subgroup $SO(2) \times SO(2) \times SO(2)$ or equivalently $U(1) \times U(1) \times U(1)$. These charges generate translations in the phases of the three complex fields

$$X = \frac{1}{\sqrt{2}}\big(\phi_1 + i\phi_2\big), \tag{3.15}$$

$$Y = \frac{1}{\sqrt{2}}\big(\phi_3 + i\phi_4\big), \tag{3.16}$$

$$Z = \frac{1}{\sqrt{2}}\big(\phi_5 + i\phi_6\big), \tag{3.17}$$

as is well known from the theory of angular momentum.[8] We can then label any product of the complex fields above by the three quantum numbers J_1, J_2 and J_3 and we will write

$$\mathcal{O}_{J_1,J_2,J_3}(x) = Tr[XXYXZXYY\ldots], \tag{3.18}$$

for a field with J_1 X's, J_2 Y's and J_3 Z's.

The conformal algebra commutes with the $SU(4)_R$ algebra, but both are part of an even larger superconformal algebra denoted $PSU(2,2|4)$. When one has a field theory with only Poincaré invariance and supersymmetry, the algebra of the generators closes to form a graded Lie algebra called a superalgebra, but when one extends Poincaré invariance to conformal invariance the algebra no longer closes. The commutators of special conformal generators and supersymmetry generators give rise to new fermionic generators called the superconformal generators and the anticommutators of superconformal generators and supercharges give rise to new bosonic generators which are the R-charges. With the R-charges, the algebra closes and we can display the superconformal algebra schematically by the matrix

$$\begin{pmatrix} P_\mu, K_\mu, M_{\mu\nu}, D & Q, \bar{S} \\ \bar{Q}, S & R \end{pmatrix}, \tag{3.19}$$

where Q represents the supercharges, S represents the superconformal charges and R represents the R-charges. The bosonic subalgebra then consists of the blocks on the diagonal, which in our case are $SO(2,4)$ and $SU(4)$.

3.2 Correlation Functions

The correlation functions we consider here are euclidian two-point functions of operators consisting of complex scalar fields such as (3.18). The combinatorics of Wick contractions

[8] For example, the function $(x \pm iy)^m$ is proportional to the spherical harmonic $Y_m^{\pm m}$ and thus carries angular momentum $\pm m$.

are rather complicated, because we are dealing with matrix fields. It can however, be efficiently captured in a matrix model correlator, which we will define below. In the end, we will be able to express a generic two-point function of complex scalars to one-loop in terms of matrix model correlators. The evaluation of these correlators is non-trivial, but as we will see in the next section, it will not be necessary to calculate them explicitly.

3.2.1 Matrix Models

The matrix model we use can be viewed as a gaussian average of a function of complex matrices M:

$$\langle f(M)\rangle_{MM} \equiv \int dMd\bar{M}f(M)e^{-Tr[M\bar{M}]}, \quad dMd\bar{M} = \prod_{a,b=1}^{N} \frac{d\mathrm{Re}M_{ab}\mathrm{Im}M_{ab}}{\pi}. \tag{3.20}$$

The integral measure is normalized so $\langle 1\rangle_{MM} = 1$. One can calculate such averages by the usual method of introducing source terms in the path integral. For example, we can write

$$\begin{aligned}\langle M_{\alpha\beta}\bar{M}_{\gamma\delta}\ldots\rangle_{MM} &= \frac{\partial}{\partial \bar{S}_{\beta\alpha}}\frac{\partial}{\partial S_{\delta\gamma}}\ldots\int dMd\bar{M}e^{-M_{ab}\bar{M}_{ba}+M_{ab}\bar{S}_{ba}+S_{ab}\bar{M}_{ba}}\Big|_{S=\bar{S}=0} \\ &= \frac{\partial}{\partial \bar{S}_{\beta\alpha}}\frac{\partial}{\partial S_{\delta\gamma}}\ldots e^{S_{ab}\bar{S}_{ba}}\Big|_{S=\bar{S}=0},\end{aligned} \tag{3.21}$$

where we completed the square

$$-M_{ab}\bar{M}_{ba} + M_{ab}\bar{S}_{ba} + S_{ab}\bar{M}_{ba} = -(M_{ab} - S_{ab})(\bar{M}_{ba} - \bar{S}_{ba}) + S_{ab}\bar{S}_{ba}, \tag{3.22}$$

and changed variables to obtain the last expression. From this, we note that any non-vanishing matrix model correlator with a given number of matrix elements should contain the same number of complex conjugated matrix elements. One trivially finds the matrix model "propagator"

$$\langle M_{\alpha\beta}\bar{M}_{\gamma\delta}\rangle_{MM} = \delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{3.23}$$

but the evaluation of correlators becomes more involved when longer strings of matrix elements are involved. For example

$$\langle Tr[M^J]Tr[\bar{M}^J]\rangle_{MM} = \frac{1}{J+1}\left(\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right), \tag{3.24}$$

with $N > J > 0$ was calculated in [24] with the help of general matrix model techniques. Fortunately, it will not be necessary to perform such calculations in the following.

Matrix models can be regarded as zero-dimensional field theories and are nice tools for extracting the combinatorics involved when one is to calculate correlation functions with trivial spacetime dependence in a matrix field theory.

3.2.2 Tree-level Correlators

The free propagator of the scalar fields in $\mathcal{N}=4$ SYM can be obtained directly from the action (3.11). It is the Green function for the laplacian operator and is given by

$$\begin{aligned}\langle 0|\phi_i^a(x)\phi_j^b(0)|0\rangle &= g_{YM}^2\delta_{ij}\delta^{ab}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2} = g_{YM}^2\delta_{ij}\delta^{ab}\int \frac{d^4p}{(2\pi)^4}e^{ipx}\int_0^\infty d\alpha e^{-\alpha p^2} \\ &= g_{YM}^2\delta_{ij}\delta^{ab}\int_0^\infty d\alpha e^{-\frac{x^2}{a\alpha}}\int \frac{d^4p}{(2\pi)^4}e^{-\alpha p^2} \\ &= \frac{g_{YM}^2}{(4\pi)^2}\delta_{ij}\delta^{ab}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-\frac{x^2}{a\alpha}} = \frac{g_{YM}^2}{4\pi^2x^2}\delta_{ij}\delta^{ab}. \end{aligned} \tag{3.25}$$

It will be more useful to use the propagators of matrix elements instead of the component fields as above. Using (3.2), and the result above we see that these can be written

$$\langle 0|(\phi_i)_{\alpha\beta}(x)(\phi_j)_{\gamma\delta}(0)|0\rangle = \frac{g_{YM}^2}{8\pi^2x^2}\delta_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{3.26}$$

If we use the Feynman gauge, the propagators of gauge fields are given by the same expressions as (3.25) and (3.26), so we can write

$$\langle 0|A^\mu_{\alpha\beta}(x)A^\nu_{\gamma\delta}(0)|0\rangle = \frac{g_{YM}^2}{8\pi^2x^2}\delta^{\mu\nu}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{3.27}$$

We will mostly be interested in correlation functions of the complex scalar fields given in (3.15)-(3.17). The free propagators are easily calculated from (3.26) and we have for example

$$\langle 0|Z_{\alpha\beta}(x)\bar{Z}_{\gamma\delta}(0)|0\rangle_0 = \frac{g_{YM}^2}{8\pi^2x^2}\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{3.28}$$

$$\langle 0|Z_{\alpha\beta}(x)Z_{\gamma\delta}(0)|0\rangle_0 = \langle 0|\bar{Z}_{\alpha\beta}(x)\bar{Z}_{\gamma\delta}(0)|0\rangle_0 = 0, \tag{3.29}$$

and similar for X and Y. From the equation above, it should be clear that any non-vanishing correlation function consisting of complex scalar fields contains an equal number of complex conjugated fields. A generic field made out of a string of complex scalars has the form (3.18), and the correlation functions we are interested in, are the two-point functions

$$\langle 0|\mathcal{O}(x)\bar{\mathcal{O}}(0)|0\rangle. \tag{3.30}$$

As an example, we consider the operator $\mathcal{O}(x) = Tr[Z^J]$. At tree-level, the two-point function of this operator and its complex conjugate taken at the origin gives J factors of the scalar propagator times a combinatorial factor. The combinatorial factor is exactly that given by the matrix model correlator (3.24), and we can write

$$\langle 0|Tr[Z^J]Tr[\bar{Z}^J]|0\rangle_0 = \left(\frac{g_{YM}^2}{8\pi^2x^2}\right)^J \langle Tr[Z^J]Tr[\bar{Z}^J]\rangle_{MM}. \tag{3.31}$$

This is also true for operators that are products of all three complex scalars. At tree-level, the two-point correlation function of operators containing an arbitrary number of X, Y, and Z can thus be written

$$\langle 0|\mathcal{O}_\alpha(x)\bar{\mathcal{O}}_\beta(0)|0\rangle_0 = \left(\frac{g_{YM}^2}{8\pi^2 x^2}\right)^J \langle \mathcal{O}_\alpha \bar{\mathcal{O}}_\beta\rangle_{MM}. \tag{3.32}$$

The combinatorics are of course more complicated when three complex scalars are involved, since then the matrix model should include three matrix fields.

3.2.3 One-loop Corrections

We now wish to calculate the one-loop radiative corrections to the two-point correlation functions above. There is a nice cancelation of the one-loop interaction terms at hand as shown in [26]. To see how it comes about, we leave the complex fields for a while and calculate two-point functions of multi-trace operators consisting of arbitrary sequences of the six real fields ϕ_i:

$$\mathcal{O}(x) = Tr[\phi_{i_1}(x)\phi_{i_2}(x)\phi_{i_3}(x)\ldots]Tr[\ldots]\ldots, \tag{3.33}$$

with a total of J fields. The two-point correlation function of such operators can be calculated with the expression

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle = \exp\Big(\frac{1}{2}\frac{\delta}{\delta\varphi}\Delta_\varphi\frac{\delta}{\delta\varphi}\Big)\mathcal{O}(x)\mathcal{O}(0)\exp^{-S_I}\Big|_{\varphi=0}, \tag{3.34}$$

where we have used φ to symbolically represent all the fields in the action and the integrals and sums in the first exponential have been suppressed.

The first order contributions to the two-point functions come from the scalar four-point interaction, the scalar self-energy, and the gluon exchange .These are shown in figure 5 for single-trace operators, where we have represented the operators as circles with J insertions to represent the trace structure. It should be noted that to get the gluon exchange and scalar self energy, the action must be expanded to second order, whereas the scalar interaction term comes from an expansion to first order. This is due to the factor of g_{YM}^2 that comes along with the propagators. The four-point interaction has two propagators more than a free diagram and one interaction vertex, whereas the scalar self-energy and gluon exchange both have three propagators more than a free diagram and two interaction vertices. The three diagrams thus each contribute with one extra factor of g_{YM}^2 in the correlation functions.

The diagrams are all divergent and need to be properly regularized, but we will not go into detail with this. Instead, we will show how the one-loop corrections to the two-point functions can be expressed in terms of matrix model vertices. We start with the scalar interaction part of the action, which can be written

$$\frac{1}{4g_{YM}^2}\int d^4x f^{abc}f^{ade}\phi_i^b\phi_j^c\phi_i^d\phi_j^e = -\frac{1}{2g_{YM}^2}\int d^4x Tr[\phi_i,\phi_j][\phi_i,\phi_j]. \tag{3.35}$$

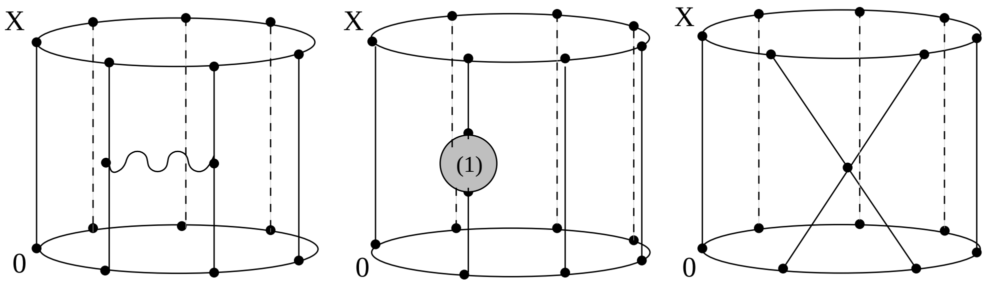

Figure 5: The three contributions to two-point correlation functions at one-loop. Gluon exchange, scalar self-energy and scalar interaction. The gluon exchange, and scalar self-energy involve two extra vertices reflecting that they correspond to a second order expansion in the interaction part of the action.

Once again, the combinatorics of the calculation become quite involved and we want to separate the correlation function into a spacetime part and a matrix model correlator. Tadpole diagrams do not contribute to the final expression, so the four fields in the interaction term should only be contracted with external fields. The one-loop contribution from the scalar interaction can then be written

$$\begin{aligned}\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle_{SI} &= \frac{1}{2g_{YM}^2}\left(\frac{g_{YM}^2}{8\pi^2 x^2}\right)^{J-2}\left(\frac{g_{YM}^2}{8\pi^2}\right)^4 \langle \mathcal{O}^+ V_{SI} \mathcal{O}^-\rangle_{MM} \int \frac{d^4z}{(z-x)^4 z^4} \\ &= -\left(\frac{g_{YM}^2}{8\pi^2 x^2}\right)^J \frac{g_{YM}^2 L}{64\pi^2}\langle \mathcal{O}^+ V_{SI} \mathcal{O}^-\rangle_{MM}, \end{aligned} \tag{3.36}$$

where in the last line we defined L to be the minus divergent integral times $x^4/2\pi^2$. The integral can be evaluated in dimensional regularization with $d = 4 - \epsilon$ and gives

$$L = \log x^{-2} - (\frac{1}{\epsilon} + \gamma + \log \pi + 2) \equiv \log(\Lambda x)^{-2}, \tag{3.37}$$

where we have introduced a constant Λ that goes to infinity as $\epsilon \to 0$. The divergent term can be canceled by an appropriate renormalization of the operators in the theory. The combinatorial factor has been written as the matrix model correlator $\langle \mathcal{O}^+ V_{SI} \mathcal{O}^-\rangle_{MM}$. To find an explicit expression for this factor, we must use the fact that the scalar fields in (3.35) can either couple to fields sitting at x or 0, and we need a way to keep track of these two possibilities. This is accomplished by introducing the matrix model fields

$$\phi_i(0) \to \phi_i^-, \qquad \phi_i(x) \to \phi_i^+, \tag{3.38}$$

with "propagators"

$$\langle (\phi_i^-)_{\alpha\beta} (\phi_j^+)_{\gamma\delta}\rangle_{MM} = \delta_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{3.39}$$

$$\langle (\phi_i^-)_{\alpha\beta} (\phi_j^-)_{\gamma\delta}\rangle_{MM} = \langle (\phi_i^+)_{\alpha\beta} (\phi_j^+)_{\gamma\delta}\rangle_{MM} = 0. \tag{3.40}$$

The matrix model operators $\mathcal{O}^+$ and $\mathcal{O}^-$ above consist only of ϕ^+'s and ϕ^-'s, respectively, and the matrix model vertex V_{SI} can be found by replacing all the fields in the interaction term by $\phi_i \to \phi_i^+ + \phi_i^-$ and collecting terms with two ϕ^+ and two ϕ^-. This yields the matrix model vertex

$$\frac{1}{2}V_{SI} =: Tr[\phi_i^+, \phi_j^-][\phi_i^+, \phi_j^-] : + : Tr[\phi_i^+, \phi_j^-][\phi_i^-, \phi_j^+] : + : Tr[\phi_i^+, \phi_j^+][\phi_i^-, \phi_j^-] : .$$

The colons denote normal ordering and means that when the vertex is placed in a correlator, the matrix model fields should only be contracted with external fields, not among themselves. This is because the tadpole diagrams do not contribute to the two-point functions. It is convenient to rewrite the second term in this vertex as

$$\begin{aligned} Tr[\phi_i^+, \phi_j^-][\phi_i^-, \phi_j^+] &= Tr\phi_i^+[\phi_j^-, [\phi_i^-, \phi_j^+]] && (3.41)\\ &= -Tr\phi_i^+[\phi_i^-, [\phi_j^+, \phi_j^-]] - Tr\phi_i^+[\phi_j^+, [\phi_j^-, \phi_i^-]] && (3.42)\\ &= -Tr[\phi_i^+\phi_i^-][\phi_j^+, \phi_j^-] - Tr[\phi_i^+, \phi_j^+][\phi_j^-, \phi_i^-], && (3.43) \end{aligned}$$

where we used the Jacobi identity and the fact that the fields can be cyclic permutated inside the trace. Inserting this, the matrix model vertex becomes

$$\frac{1}{2}V_{SI} =: V_D : + : V_F : + : V_K :, \qquad (3.44)$$

where

$$\begin{aligned} V_D &= -Tr[\phi_i^+, \phi_i^-][\phi_j^+, \phi_j^-], && (3.45)\\ V_F &= 2Tr[\phi_i^+, \phi_j^+][\phi_i^-, \phi_j^-], && (3.46)\\ V_K &= Tr[\phi_i^+, \phi_j^-][\phi_i^+, \phi_j^-]. && (3.47) \end{aligned}$$

This form of the vertex is practical, because it has been split in parts that couple to the symmetric, antisymmetric, and trace part of operators as we now explain. Consider the linear combination of operators

$$\mathcal{O}_k = C_{i_1 i_2 \dots i_k} Tr[\phi_{i_1}\phi_{i_1}\phi_{i_2}\dots\phi_{i_k}]. \qquad (3.48)$$

When the matrix model vertex V_{SI} is contracted with this operator, we can take all the fields to be minus valued and contract them with the plus valued fields in V_{SI}. We then see that V_D only couples to the symmetric part, V_F couples to the antisymmetric part, and V_K couples to the trace part. Here, trace refers to a contraction of any two indices. As an example, one can easily verify that the Konishi operator $\mathcal{K} = Tr[\phi_i\phi_i]$, couples to V_K and V_D, but not to V_F.

One can perform similar calculations with the gluon exchange term and the scalar self interaction. The result is [26]

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle_{SE} = -\Big(\frac{g_{YM}^2}{8\pi^2 x^2}\Big)^J \frac{g_{YM}^2(L+1)}{8\pi^2}\langle \mathcal{O}^+ : V_{SE} : \mathcal{O}^-\rangle_{MM}, \qquad (3.49)$$

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle_{GE} = -\Big(\frac{g_{YM}^2}{8\pi^2 x^2}\Big)^J \frac{g_{YM}^2(L+2)}{32\pi^2}\langle \mathcal{O}^+ : V_{GE} : \mathcal{O}^-\rangle_{MM}, \qquad (3.50)$$

where the matrix model vertices are given by

$$V_{SE} = Tr(\phi_i^-)Tr(\phi_i^+) - NTr(\phi_i^-\phi_i^+), \tag{3.51}$$
$$V_{GE} = V_D, \tag{3.52}$$

The complete one-loop contribution to the correlation function is then given by the sum of (3.36), (3.49), and (3.50):

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle_{1-loop} = -\Big(\frac{g_{YM}^2}{8\pi^2 x^2}\Big)^J \Big[\frac{g_{YM}^2(L+1)}{8\pi^2}\langle \mathcal{O}^+\Big(\frac{1}{2} : V_D : + : V_{SE} :\Big)\mathcal{O}^-\rangle_{MM} + \frac{g_{YM}^2 L}{32\pi^2}\langle \mathcal{O}^+\big(: V_F : + : V_K : \big)\mathcal{O}^-\rangle_{MM}\Big]. \tag{3.53}$$

The remarkable thing here is that the matrix model correlator involving V_D and V_{SE} vanishes. To show this, we first note that we can write the vertices without the normal ordering symbols by just subtracting the terms we get from self-contractions among the fields in a vertex:

$$: -\frac{1}{2}Tr[\phi_i^+,\phi_i^-][\phi_j^+,\phi_j^-] := -\frac{1}{2}Tr[\phi_i^+,\phi_i^-][\phi_j^+,\phi_j^-] + 2NTr(\phi_i^-\phi_i^+) - 2Tr(\phi_i^-)Tr(\phi_i^+),$$

and the matrix model correlator in the first line of (3.53) can thus be written

$$\langle \mathcal{O}^+\Big(-\frac{1}{2}[\phi_i^+,\phi_i^-][\phi_j^+,\phi_j^-] + NTr(\phi_i^-\phi_i^+) - Tr(\phi_i^-)Tr(\phi_i^+)\Big)\mathcal{O}^-\rangle_{MM}. \tag{3.54}$$

We concentrate on the first ϕ_i^+ in this vertex. It must either be contracted with external ϕ^-'s or with one of the ϕ^- appearing in the vertex itself. If we consider an arbitrary trace of k scalars $\mathcal{O}^- = Tr[\phi_{i_1}^-\phi_{i_2}^-\phi_{i_3}^-\ldots\phi_{i_k}^-]$ and sum all possible contractions with this field, we get

$$\begin{aligned}
Tr[\phi_i^+,\phi_i^-][\phi_j^+,\phi_j^-]\circ Tr[\phi_{i_1}^-\phi_{i_2}^-\phi_{i_3}^-\ldots\phi_{i_k}^-]&\\
=&Tr\big([\phi_{i_1}^-,[\phi_j^+,\phi_j^-]]\phi_{i_2}^-\phi_{i_3}^-\ldots\phi_{i_k}^-\big) + Tr\big(\phi_{i_1}^-[\phi_{i_2}^-,[\phi_j^+,\phi_j^-]]\phi_{i_3}^-\ldots\phi_{i_k}^-\big)\\
&+ Tr\big(\phi_{i_1}^-\phi_{i_2}^-[\phi_{i_3}^-,[\phi_j^+,\phi_j^-]]\ldots\phi_{i_k}^-\big) + \ldots + Tr\big(\phi_{i_1}^-\phi_{i_2}^-\phi_{i_3}^-\ldots\phi_{i_{k-1}}^-[\phi_{i_k}^-,[\phi_j^+,\phi_j^-]]\big)\\
=&-Tr\big([\phi_j^+,\phi_j^-]\phi_{i_1}^-\phi_{i_2}^-\phi_{i_3}^-\ldots\phi_{i_k}^-\big) + Tr\big(\phi_{i_1}^-\phi_{i_2}^-\phi_{i_3}^-\ldots\phi_{i_k}^-[\phi_j^+,\phi_j^-]\big) = 0,
\end{aligned}$$

where we used the cyclicity of the trace and that terms in the sum cancel pairwise. Furthermore, contracting the first ϕ_i^+ in (3.54) with the ϕ^-'s in the vertex itself results in terms that cancel with the remaining quadratic terms in (3.54). The only vertices that contribute to one-loop correlation functions are thus V_F and V_K, and we get the final result

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle_{1-loop} = -\Big(\frac{g_{YM}^2}{8\pi^2 x^2}\Big)^J \frac{g_{YM}^2 L}{32\pi^2}\langle \mathcal{O}^+\big(: V_F : + : V_K : \big)\mathcal{O}^-\rangle_{MM}. \tag{3.55}$$

Since V_F and V_K only couple to the antisymmetric and trace part of the tensor operators in (3.48), correlation functions of operators that are symmetric and traceless do not receive one-loop radiative corrections. Such operators are called chiral primary or 1/2 BPS and have vanishing anomalous dimension. Examples of such operators are

$$Tr[X^J], \qquad Tr[Y^J], \qquad Tr[Z^J], \qquad Tr[\phi_i\phi_j] - \frac{1}{6}\delta_{ij}Tr[\phi_k\phi_k]. \tag{3.56}$$

In fact, we will use $Tr[Z^J]$ as a "groundstate" when we explicitly calculate the anomalous dimension of a certain class of operators in the next section.

We now return to operators that are words of the complex scalar fields X, Y, and Z and consider the correlators $\langle 0|\mathcal{O}(x)\bar{\mathcal{O}}(0)|0\rangle$. Such operators are traceless in the sense used above and we can neglect the matrix model vertex V_K. This can easily be seen if we express the vertex in terms of the complex scalars:

$$\begin{aligned}V_K =&[\phi_i^+,\phi_j^-][\phi_i^+,\phi_j^-]\\ =&\frac{1}{4}Tr\big([X^+ + \bar{X}^+, X^- + \bar{X}^-]^2 - [X^+ + \bar{X}^+, X^- - \bar{X}^-]^2 + \dots\big)\\ =&Tr\big([X^+,X^-][X^+,\bar{X}^-] + [X^+,\bar{X}^-][\bar{X}^+,X^-]\\ &+ [\bar{X}^+,\bar{X}^-][\bar{X}^+,X^-] + [\bar{X}^+,\bar{X}^-][X^+,X^-] + \dots\big).\end{aligned} \tag{3.57}$$

A non-vanishing matrix model correlator should contain an equal number of barred and un-barred fields, so the first and third term will not contribute. In addition, the correlators only contain un-barred fields at x (which is translated to X^+ in the matrix model correlator) and therefore, the second and fourth terms cannot be completely contracted with external any fields in the correlator. We are then left with V_F, which we will also express in terms of complex scalars:

$$\begin{aligned}V_F =&2Tr[\phi_i^+,\phi_j^+][\phi_i^-,\phi_j^-]\\ =&-2Tr[X^+ + \bar{X}^+, X^+ - \bar{X}^+][X^- + \bar{X}^-, X^- - \bar{X}^-]\\ &+ 2Tr[X^+ + \bar{X}^+, Y^+ + \bar{Y}^+][X^- + \bar{X}^-, Y^- + \bar{Y}^-] - \dots\end{aligned} \tag{3.58}$$

Again, since we should contract this with operators in the matrix model correlators that only contain plus valued X, Y, and Z and minus valued $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, we throw away terms in the vertex that contain minus valued X, Y, and Z or plus valued $\bar{X}$, $\bar{Y}$, and $\bar{Z}$. We are then left with

$$V_F = 4Tr\big([X^+,Y^+][\bar{X}^-,\bar{Y}^-] + [Y^+,Z^+][\bar{Y}^-,\bar{Z}^-] + [Z^+,X^+][\bar{Z}^-,\bar{X}^-]\big). \tag{3.59}$$

In the following, we suppress the $+/-$ superscripts, since we know that they go with un-barred/barred fields.

The tree-level and one-loop contributions to correlation functions of complex scalars share a common factor of $(g_{YM}^2/8\pi^2x^2)^J$. We can get rid of this by rescaling operators containing J complex scalar fields by a factor of $(g_{YM}^2/8\pi^2)^{J/2}$, so the tree-level part is zeroth order in g_{YM}^2, and the one-loop part is first order in g_{YM}^2. We can then summarize the evaluation of two-point correlation function of complex scalars to one-loop: Any multi-trace operator O_α involving the three complex scalars X, Y, and Z, with tree-level conformal dimension J, is given to one-loop by

$$\langle \mathcal{O}_\alpha(x)\bar{\mathcal{O}}_\beta(0)\rangle = \frac{1}{x^{2J}}\Big(S_{\alpha\beta} + T_{\alpha\beta}\log(kx)^{-2}\Big), \tag{3.60}$$

where k is a constant introduced to make the argument of the logarithm dimensionless, $S_{\alpha\beta}$ and $T_{\alpha\beta}$ are the matrix model correlators

$$S_{\alpha\beta} = \langle \mathcal{O}_\alpha \bar{\mathcal{O}}_\beta\rangle_{MM}, \qquad T_{\alpha\beta} = \langle \mathcal{O}_\alpha H \bar{\mathcal{O}}_\beta\rangle_{MM}, \tag{3.61}$$

and H is given by the interaction vertex

$$H = -\frac{g_{YM}^2}{8\pi^2} : \Big(Tr[X,Y][\bar{X},\bar{Y}] + Tr[Y,Z][\bar{Y},\bar{Z}] + Tr[Z,X][\bar{Z},\bar{X}]\Big) : . \tag{3.62}$$

3.3 The Dilatation Operator

We will now show how to use the general form of correlation functions in conformal field theory to relate the dilatation operator with the effective vertex (3.62) obtained in perturbation theory.

In the last section, it was shown that two-point correlation functions of fields with definite conformal dimension have the generic form

$$\langle \mathcal{O}_{\Delta_1}(x_1)\bar{\mathcal{O}}_{\Delta_1}(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}, \tag{3.63}$$

where C_{12} is zero unless $\Delta_1 = \Delta_2$. The fields all have tree-level conformal dimension $\Delta = J_1 + J_2 + J_3 \equiv J$, but when quantum corrections are included, the operators will mix and are no longer have definite conformal dimension. To find a set of operators that have a well-defined conformal dimension when one-loop radiative corrections are included, we need to consider the set of multi-trace operators

$$\mathcal{O}_\alpha = Tr[XXYXZ\ldots]Tr[ZZYZX\ldots]Tr[XYYYX\ldots]\ldots, \tag{3.64}$$

with tree-level conformal dimension J. It is not enough to just consider the single-trace operators (3.18) since in general the eigenstates of the dilatation operator will be linear combinations of both single and multi-trace operators.

We are looking for a basis transformation to a set of fields $\mathcal{O}'_A$ that carry a definite scaling dimension at one-loop. We denote the anomalous piece of the dilatation operator by $\mathcal{D}$, and have $\mathcal{D}\mathcal{O}'_A = \Delta_A \mathcal{O}'_A$, where Δ_A is the anomalous dimension associated with $\mathcal{O}'_A$. Expressing the fields (3.64) in this basis gives

$$\mathcal{O}_\alpha = V_{\alpha A}\mathcal{O}'_A, \tag{3.65}$$

and we have

$$\mathcal{D}\mathcal{O}_\alpha = \Delta_A V_{\alpha A}\mathcal{O}'_A = V_{\alpha A}\Delta_A V^{-1}_{A\beta}\mathcal{O}_\beta \equiv \mathcal{D}_{\alpha\beta}\mathcal{O}_\beta, \tag{3.66}$$

where we have defined the anomalous dimension matrix $\mathcal{D}_{\alpha\beta}$.

We will now assume that the one-loop corrections are small and write $\Delta = J + \Delta_A$, with $\Delta_A \ll 1$. Using equation (3.63) with the fields $\mathcal{O}'_A$ and expanding $|x|^{-2\Delta_A}$ in the small exponent then give

$$\langle \mathcal{O}'_A(x)\bar{\mathcal{O}}'_B(0)\rangle = \frac{\delta_{AB}C_A}{|x|^{2J}}\Big(1 + \Delta_A \log|kx|^{-2}\Big), \tag{3.67}$$

where k is just an arbitrary constant extracted from the normalization to make the argument of the logarithm dimensionless. Using this expression and (3.65), we get for a generic two-point function:

$$\begin{aligned}
\langle \mathcal{O}_\alpha(x)\bar{\mathcal{O}}_\beta(0)\rangle &= V_{\alpha A}V^*_{\beta B}\langle \mathcal{O}'_A(x)\bar{\mathcal{O}}'_B(0)\rangle && (3.68)\\
&= V_{\alpha A}V^*_{\beta B}\frac{\delta_{AB}C_A}{|x|^{2J}}\Big[1 + \Delta_A \log|kx|^{-2}\Big] && (3.69)\\
&= \frac{1}{|x|^{2J}}\Big[(VCV^\dagger)_{\alpha\beta} + (VC\Delta V^\dagger)_{\alpha\beta}\log|kx|^{-2}\Big], && (3.70)
\end{aligned}$$

where we defined the diagonal matrices $C_{AB} = \delta_{AB}C_A$ and $\Delta_{AB} = \delta_{AB}\Delta_A$. This should coincide with the result obtained using perturbation theory and Feynman diagrams (3.60). Comparing the two, we see that we can identify the matrices

$$\begin{aligned}
S_{\alpha\beta} &= \langle \mathcal{O}_\alpha\bar{\mathcal{O}}_\beta\rangle_{MM} = (VCV^\dagger)_{\alpha\beta}, && (3.71)\\
T_{\alpha\beta} &= \langle \mathcal{O}_\alpha H\bar{\mathcal{O}}_\beta\rangle_{MM} = (VC\Delta V^\dagger)_{\alpha\beta}, && (3.72)
\end{aligned}$$

and we get

$$(TS^{-1})_{\alpha\beta} = (VC\Delta V^\dagger V^{\dagger -1}C^{-1}V^{-1})_{\alpha\beta} = (V\Delta V^{-1})_{\alpha\beta} = \mathcal{D}_{\alpha\beta}, \tag{3.73}$$

since the diagonal matrices Δ and C commute.

Now, H is given by

$$H = -\frac{g_{YM}^2}{8\pi^2} : \Big(Tr[X,Y][\bar{X},\bar{Y}] + Tr[Y,Z][\bar{Y},\bar{Z}] + Tr[Z,X][\bar{Z},\bar{X}]\Big) :, \tag{3.74}$$

and if we contract the barred fields of H with any unbarred fields in the multi-trace operator (3.64), we get an operator of the same kind. This means that we can write

$$H \circ \mathcal{O}_\alpha = H_{\alpha\beta}\mathcal{O}_\beta, \tag{3.75}$$

or

$$T_{\alpha\beta} = \langle \mathcal{O}_\alpha H\bar{\mathcal{O}}_\beta\rangle_{MM} = H_{\alpha\gamma}\langle \mathcal{O}_\gamma\bar{\mathcal{O}}_\beta\rangle_{MM} = H_{\alpha\gamma}S_{\gamma\beta}, \tag{3.76}$$

and therefore we have the identity

$$H_{\alpha\beta} = T_{\alpha\gamma}S^{-1}_{\gamma\beta} = \mathcal{D}_{\alpha\beta}. \tag{3.77}$$

This final result is very relieving. We do not have to delve into an involved evaluation of the matrix model correlators $S_{\alpha\beta}$ and $T_{\alpha\beta}$ to find anomalous dimensions. Instead, we need to diagonalize $H_{\alpha\beta}$ on the set of multi-trace operators (3.64). This is not an easy task, however, but as we will show, the problem is simplified a lot in the planar (large N) limit, where we can map the dilatation operator to a Heisenberg spin chain.

3.3.1 The Planar Limit

In order to calculate the anomalous dimension matrix, we need to consider contractions of a generic field with H.

To see how this works, consider the first term appearing in H contracted with a generic single-trace operator $\mathcal{O}_\alpha$ inside the correlator $\langle \mathcal{O}_\alpha H \bar{\mathcal{O}}_\beta \rangle_{MM}$

$$Tr[X,Y][\bar{X},\bar{Y}] \circ Tr[XXYXZ\ldots Z], \tag{3.78}$$

or, if we write out the indices

$$[X_{ab}Y_{bc} - Y_{ab}X_{bc}][\bar{X}_{cd}\bar{Y}_{da} - \bar{Y}_{cd}\bar{X}_{da}] \circ [X_{ij}X_{jk}Y_{kl}X_{lm}Z_{mn}\ldots Z_{.i}]. \tag{3.79}$$

We should make all possible contractions and if we remember the simple form of the propagators $\langle X_{ab}\bar{X}_{cd}\rangle_{MM} = \delta_{ad}\delta_{bc}$ and similar for Y and Z, it should be clear that we can obtain the contractions by mapping

$$\bar{X}_{ab} \to \frac{d}{dX_{ba}}, \qquad \bar{Y}_{ab} \to \frac{d}{dY_{ba}}, \qquad \bar{Z}_{ab} \to \frac{d}{dZ_{ba}}, \tag{3.80}$$

and letting them act directly on the zero-dimensional matrix operators. Using this picture, we do not have to worry about matrix model correlators and Wick contractions any more. We simply obtain the anomalous dimension matrix $\mathcal{D}_{\alpha\beta}$ by acting directly on matrix model operators $\mathcal{O}_\alpha$ with H written as a differential operator.

We now consider the action of the XY term in H on a single-trace operator $\mathcal{O} = Tr[XAYB]$, where A and B stand for some product of the three complex scalars. We have singled out arbitrary X and Y to see what the contribution from contractions with such fields will be. We get

$$\begin{aligned} Tr[X,Y][\bar{X},\bar{Y}] \circ' Tr[XAYB] &= [X,Y]_{ab}[\bar{X}_{bc}\bar{Y}_{ca} - \bar{Y}_{bc}\bar{X}_{ca}] \circ' [X_{ij}A_{jk}Y_{kl}B_{li}] \\ &= [X,Y]_{ab}[A_{ba}B_{cc} - A_{cc}B_{ba}] \\ &= -Tr[B]Tr[Y,X]A - Tr[A]Tr[X,Y]B, \end{aligned} \tag{3.81}$$

where the primed circle implies that we are only considering the contribution from the explicitly appearing X and Y. If X and Y are nearest neighbors, then either A or B are identity matrices, and we get a factor of N times a single-trace operator. In the planar limit of the gauge theory where N is taken to infinity, the terms that do not have a factor of N are sub-leading and we thus see that, when acting with $Tr[X,Y][\bar{X},\bar{Y}]$, the leading contributions are those, where the contracted scalars are nearest neighbors in the operator. In that case we get minus N times the original field plus N times the original field where X and Y have switched positions. The same argument holds true for the other two terms in H and we can thus write

$$\mathcal{D}_1^{Planar} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{J}(1 - P_{i,i+1}), \tag{3.82}$$

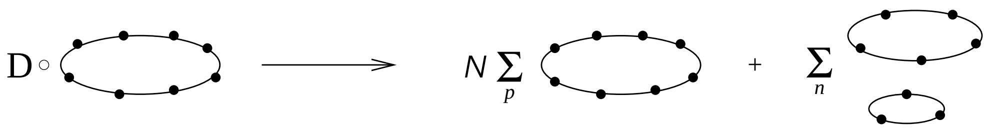

Figure 6: The action of the dilatation operator on a single-trace operator results in both single-trace operators and two-trace operators, but only the single-trace operators are multiplied by a factor of N. The dots represent either X, Y, or Z and the sum in front of the single-trace is over all possible nearest neighbor contractions. The sum in front of the two-trace is over all possible ways of constructing the two-trace operator.

where we have introduced the 't Hooft coupling $\lambda = N g_{YM}^2$ and the permutation operator $P_{i,i+1}$ that permutes the scalar fields at sites i and $i+1$.

From (3.81), we observe that in general, the action of H on a single-trace operator will result in both two-trace operators and single-trace operators, but only the single-trace operators will carry a factor of N. This is illustrated in figure 6. Conversely, when we act with the dilatation operator on a two-trace operator, we get both one, two, and three-trace operators, but only the resulting two-trace operators carry a factor of N. The sector of single-trace operators thus closes in the planar limit and the action of $\mathcal{D}_1^{Planar}$ on single-trace operators will only give back new single-trace operators with the number of each of the three complex scalars conserved.

The theme of next section will be the diagonalization of (3.82), but first we will show how the dilatation operator is affected under certain deformations of $\mathcal{N} = 4$ SYM.

3.4 Marginal Deformations of $\mathcal{N} = 4$ SYM

It is interesting to examine whether the AdS/CFT correspondence can be generalized to theories with less symmetry than $\mathcal{N} = 4$ SYM. Since, we have seen that the duality is intimately connected to the conformal symmetry of the field theory, we will only be concerned with deformations of $\mathcal{N} = 4$ SYM that do not destroy the conformal symmetry.[9] This class of deformations are called marginal deformations and are characterized by leaving the β-function invariant. It is most convenient to use the $\mathcal{N} = 1$ superspace formalism [23, 25], in which the lagrangian of $\mathcal{N} = 4$ SYM is described by a spinorial superfield strength W^α, a vector superfield V, and three chiral superfields Φ_i. The three scalars X, Y, and Z are the lowest components of the three chiral superfields and their coupling is included in the superpotential

$$W = Tr[\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2]. \tag{3.83}$$

[9] It would of course be extremely interesting to generalize the duality to non-conformal field theories, since QCD does not have conformal invariance at the quantum level. There have been some attempts at this [19], but it has proven hard and is beyond the scope of this thesis.

The scalar coupling term in the lagrangian can be obtained from this expression by taking

$$\mathcal{L}_{SC} = Tr\left|\frac{\partial W}{\partial \Phi_1}\right|^2 + Tr\left|\frac{\partial W}{\partial \Phi_2}\right|^2 + Tr\left|\frac{\partial W}{\partial \Phi_3}\right|^2. \tag{3.84}$$

Only the lowest components of the superfields will survive in this expression so the result can be obtained by replacing the superfields in (3.84) and (3.83) by X, Y, and Z, which yields

$$\mathcal{L}_{SC} = Tr\Big[[X,Y][\bar{X},\bar{Y}] + [Y,Z][\bar{Y},\bar{Z}] + [Z,X][\bar{Z},\bar{X}]\Big]. \tag{3.85}$$

A class of marginal deformations was constructed by Leigh and Strassler [16] and is referred to as Leigh-Strassler or β-deformations. A β-deformed version of $\mathcal{N}=4$ SYM contains the superpotential

$$W_{LS} = ATr[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2] + hTr[\Phi_1^3 + \Phi_2^3 + \Phi_3^3], \tag{3.86}$$

where h and β are complex parameters and A is a function that can be determined by the requirement of conformal invariance [27]. We will consider only the simplest of these where $h=0$, β is real and $A=1$. The $\mathcal{N}=4$ supersymmetry is then broken to $\mathcal{N}=1$, while preserving the Cartan subalgebra e.i. $SU(4)_R \to U(1)\times U(1)\times U(1)$. Neither the $SU(4)$ symmetry of the undeformed theory, nor the complete $U(1)^3$ symmetry of the deformed theory are manifest in the superfield formulation, but the symmetries will explicitly show themselves when we write the interaction in terms of complex scalars. Denoting the real part of β by γ, the deformed superpotential can then be written

$$W_\gamma = Tr[e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2]. \tag{3.87}$$

Until now, we have used the charges that generate translations in the phase of each of the three complex scalars as a base of the Cartan subalgebra. However, the $U(1)$ R-symmetry of the deformed theory is mixed in these three $U(1)$ generators. Therefore, we will make a change of base so one of the $U(1)$ symmetries act by $\Phi_i \to e^{i\varphi_0}\Phi_i$ for $i\in\{1,2,3\}$. The two remaining $U(1)$ symmetries are then global non-R-symmetries of the theory. Their action on the fields are

$$1: \qquad (\Phi_1,\Phi_2,\Phi_3) \to (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3), \tag{3.88}$$

$$2: \qquad (\Phi_1,\Phi_2,\Phi_3) \to (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3), \tag{3.89}$$

and are easily seen to leave the superpotential (3.87) invariant. These symmetries will play an important role in the identification of the gravity dual of the deformed theory.

The scalar interaction term can again be computed using (3.84) with X, Y, and Z instead of the chiral superfields, and the result is

$$\begin{aligned}\mathcal{L}_{SC_\gamma} = Tr\Big[&YZ\bar{Z}\bar{Y} + ZY\bar{Y}\bar{Z} - e^{2\pi i\gamma}YZ\bar{Y}\bar{Z} - e^{-2\pi i\gamma}ZY\bar{Z}\bar{Y}\\ &+ZX\bar{X}\bar{Z} + XZ\bar{Z}\bar{X} - e^{2\pi i\gamma}ZX\bar{Z}\bar{X} - e^{-2\pi i\gamma}XZ\bar{X}\bar{Z}\\ &+XY\bar{Y}\bar{X} + YX\bar{X}\bar{Y} - e^{2\pi i\gamma}XY\bar{X}\bar{Y} - e^{-2\pi i\gamma}YX\bar{Y}\bar{X}\Big].\end{aligned} \tag{3.90}$$

Note that this lagrangian has a Z_3 symmetry that cyclic permutes the three fields and is thus invariant under the transformation $X \to Y$, $Y \to Z$, $Z \to X$. A non-cyclic permutation of the fields (for example $X \leftrightarrow Y$) has to be accompanied by $\gamma \to -\gamma$ to leave the lagrangian invariant.

The one-loop dilatation operator can be obtained from the lagrangian (3.90). In fact following the procedure of the last two subsections, we simply have to replace the barred fields by derivatives. As in the undeformed case, the action on a given operator only gives terms involving a factor of N, when the contractions are taken with nearest neighbor fields. Once again, we only keep these terms in the planar limit, but since there is a different phase factor associated with each term, the planar dilatation operator cannot be written as simple as was done in equation (3.82). Instead, we define the states $|1\rangle_i = |X\rangle_i$, $|2\rangle_i = |Y\rangle_i$, and $|3\rangle_i = |Z\rangle_i$ associated with site i in the operator. If we then introduce the operator E^i_{kl} with the action $E^i_{kl}|m\rangle_i = |k\rangle_i \delta_{lm}$, we can write the planar dilatation operator

$$\mathcal{D}_\gamma^{Planar} = \frac{\lambda}{8\pi^2} \sum_{i=1}^{J} H_i, \tag{3.91}$$

where

$$\begin{aligned} H_i = E^i_{22}E^{i+1}_{33} + E^i_{33}E^{i+1}_{22} - e^{2\pi i\gamma}E^i_{23}E^{i+1}_{32} - e^{-2\pi i\gamma}E^i_{32}E^{i+1}_{23} \\ +E^i_{33}E^{i+1}_{11} + E^i_{11}E^{i+1}_{33} - e^{2\pi i\gamma}E^i_{31}E^{i+1}_{13} - e^{-2\pi i\gamma}E^i_{13}E^{i+1}_{31} \\ +E^i_{11}E^{i+1}_{22} + E^i_{22}E^{i+1}_{11} - e^{2\pi i\gamma}E^i_{12}E^{i+1}_{21} - e^{-2\pi i\gamma}E^i_{21}E^{i+1}_{12}. \end{aligned} \tag{3.92}$$

Again, we identify the sites $i = 1$ and $i = (J + 1)$, since we are acting on a trace of fields.

In section 5.4 we will give the gravity dual of this deformation and calculate the energy of strings in the deformed background. Since marginal deformations maintain conformal invariance, the AdS part of the gravity dual is not changed. The deformation acts on the metric of S^5 reflecting that we have broken the $SU(4)$ R-symmetry. The deformed superpotential (3.87) has an $U(1) \times U(1)$ global symmetry in addition to the remaining $U(1)$ R-symmetry and the deformed metric should thus contain a two-torus corresponding to this symmetry. This is what Lunin and Maldacena exploited in [15] to find the gravity dual of (3.87).

4 Spin Chains

This section will be devoted to diagonalizing the one-loop planar dilatation operator. To do this, we turn to a seemingly completely different problem, namely the Heisenberg spin chain. The hamiltonian for this problem is integrable and was diagonalized in 1931 by Bethe. It turns out that the one-loop planar dilatation operator can be identified with the Heisenberg hamiltonian, if we at the same time map the set of single-trace operators to cyclic spin chains.

We start by shortly reviewing the Heisenberg spin chain and how integrability allows one to diagonalize the spin chain hamiltonian exactly using Bethe's ansatz for the position space wavefunction. The problem is then shown to be equivalent to the $SU(2)$ sector of single-trace operators and anomalous dimensions of these are found in the "long" operator limit (large R-charge). The discussion is then generalized to the $SU(3)$ sector, which is a bit more involved due to the presence of different "flavors" in the excitation spectrum. Finally, we turn to the β-deformed spin chain which, by a certain change of base, can be mapped to the undeformed spin chain with twisted boundary conditions. The deformed spin chain will be analyzed in the $SU(2)$ sector and the results heuristically generalized to the $SU(3)$ sector.

4.1 The Heisenberg Model

In solid state systems with localized particles carrying magnetic moments, one can capture the basic properties with an effective hamiltonian known as the Heisenberg model [28]

$$H_{Heisenberg} = -\sum_{ij} \varepsilon_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \tag{4.1}$$

where $\boldsymbol{\sigma}_i$ is the pauli matrix acting on the particle at lattice site i. If one assumes that the interaction is short-ranged and constant, we can write $\varepsilon_{ij} = J_0 C_{ij}$, where C_{ij} is 1 if i and j are nearest neighbors and zero otherwise. We then observe that if $\varepsilon_0 < 0$, it is energetically favorable for the particles to become antiparallel, which gives an antiferromagnetic groundstate, whereas if $\varepsilon_0 > 0$, the particles tend to align their spins and we get a ferromagnetic groundstate. The hamiltonian of a one-dimensional ferromagnetic system, with J lattice sites and the energy of the groundstate set to zero, can then be written

$$H_{Sc} = \frac{\varepsilon_0}{2} \sum_{i=1}^{J} (1 - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+1}) = \varepsilon_0 \sum_{i=1}^{J} (1 - P_{i,i+1}), \tag{4.2}$$

and we will refer to this system as an $SU(2)$ spin chain. The last equality simply follows from the fact that the two operators have the same action on the four states $|\uparrow\rangle_i \otimes |\uparrow\rangle_{i+1}$, $|\downarrow\rangle_i \otimes |\downarrow\rangle_{i+1}$, $|\downarrow\rangle_i \otimes |\uparrow\rangle_{i+1}$ and $|\uparrow\rangle_i \otimes |\downarrow\rangle_{i+1}$. We use periodic boundary conditions so $P_{J,J+1} = P_{J,1}$.

The set of states we consider have the generic form $|\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow \ldots\rangle$, containing a total of J sites with M up-spins and $M \leq J/2$. If we take the ground state to be all down-spins, we can think of the up-spins as excitations called magnons or spinwaves in the terminology of solid state physics.[10]

The hamiltonian (4.2) can be shown to belong to a family of J commuting operators and is integrable (see appendix B). This is a term borrowed from classical systems where all solutions to a system with J degrees of freedom can be classified if one can obtain J integrals of motion. In a quantum system with J commuting observables, we can assign J quantum numbers to the eigenstates of the hamiltonian. Finding the eigenenergies for

[10]The condition of $M \leq J/2$ is not an actual restriction, since if $M > J/2$, we would just choose the groundstate to be all up-spins and the down-spins would then constitute the excitations.

a finite spin chain comes down to diagonalizing a $2^J \times 2^J$ matrix which is a problem accessible by computer, but in the limit where J becomes large, only analytic methods work and integrability ensures that the eigenenergies can be found.

The integrability of the spin chain is implied by the existence of an R-matrix $R(u)$ depending on a spectral parameter u and satisfying a certain Yang-Baxter relation [29]. The R-matrix fulfils

$$R(u)|_{u=u_0} = P, \qquad P\frac{d}{du}R(u)|_{u=u_0} = H, \tag{4.3}$$

where P is the permutation operator and u_0 is defined by the first of these equations. The rest of the conserved charges can be constructed from the R-matrix in a similar manner.[11]

4.1.1 The Bethe Ansatz

The eigenvalues of the hamiltonian (4.2) can be found using a rigorous approach known as the algebraic Bethe ansatz as shown in appendix B. However, the procedure is rather mathematical and it will be more useful to apply the original coordinate ansatz used by Bethe, which gives a better physical intuition for the interacting magnons. This approach was reviewed and developed beyond the one loop $SU(2)$ spin chain in [30].

As an example, we will diagonalize the one magnon sector. We denote a state with one spin-up located at site x by $|x\rangle$. The state

$$|1, J; p\rangle = \sum_{x=1}^{J} e^{ipx}|x\rangle, \tag{4.4}$$

is then seen to be diagonal under the action of H_{Sc}

$$H_{Sc}|1, J; p\rangle = \varepsilon_0 \sum_{x=1}^{J} e^{ipx}\Big(2|x\rangle - |x+1\rangle - |x-1\rangle\Big) \tag{4.5}$$

$$= \varepsilon_0 \sum_{x=1}^{J} (2 - e^{-ip} - e^{ip})e^{ipx}|x\rangle = 4\varepsilon_0 \sin^2(\frac{p}{2})|1, J; p\rangle. \tag{4.6}$$

In addition, the periodic boundary conditions give $p = \frac{2\pi n}{J}$, $n \in \mathbb{Z}$, since we require $e^{ip(x+J)} = e^{ipx}$.

The state (4.4) is a Fourier transformation of the coordinate state $|x\rangle$, and we can thus think of the magnon as a propagating particle with momentum p. One might suspect that this can be generalized to an M-magnon state, which should then be an M-fold Fourier transform of the coordinate state $|x_1 x_2 \dots x_M\rangle$ where the x_i denotes the position of the up-spins. Such a state should then be characterized by M momentum variables and simply describes M particles propagating freely along the spin chain. However, one can easily

[11] This definition of the R-matrix is the most common in the literature, but note that it differs from the one used in appendix B.

verify that for example the fourier transform of the state $|x_1x_2\rangle$ does not diagonalize the hamiltonian, so this idea is to simple. The reason is that the magnons undergo scattering when they pass each other and we thus need to consider a wavefunction which incorporates this. In the context of scattering, integrability implies that scattering events are non-diffractive, meaning that the momenta involved are individually conserved in the process.

To diagonalize the two-magnon sector, we again make a change of basis to momentum space, but now with an undetermined wavefunction $\psi_{p_1p_2}(x_1,x_2)$:

$$|2,J;p_1,p_2\rangle = \sum_{1\le x_1<x_2\le J} \psi_{p_1p_2}(x_1,x_2)|x_1x_2\rangle. \tag{4.7}$$

Inserting this into the Schrödinger equation gives the two equations

$$x_2 = x_1+1: \quad \frac{E}{\varepsilon_0}\psi_{p_1p_2}(x_1,x_2) = 2\psi_{p_1p_2}(x_1,x_2) - \psi_{p_1p_2}(x_1-1,x_2) - \psi_{p_1p_2}(x_1,x_2+1), \tag{4.8}$$

$$\begin{aligned} x_2 > x_1+1: \quad \frac{E}{\varepsilon_0}\psi_{p_1p_2}(x_1,x_2) &= 2\psi_{p_1p_2}(x_1,x_2) - \psi_{p_1p_2}(x_1-1,x_2) - \psi_{p_1p_2}(x_1+1,x_2) \\ &\quad +2\psi_{p_1p_2}(x_1,x_2) - \psi_{p_1p_2}(x_1,x_2-1) - \psi_{p_1p_2}(x_1,x_2+1), \end{aligned} \tag{4.9}$$

and these are solved by the Bethe ansatz for the wavefunction:

$$\psi_{p_1p_2}(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} + S(p_2,p_1)e^{i(p_2x_1+p_1x_2)}. \tag{4.10}$$

If both momentum variables are real, we can interpret the Bethe ansatz (4.10) as a superposition of an in, and an out state. The idea is that the two magnons should either freely propagate down the lattice or exchange momenta through scattering. The amplitude for scattering is then given by the S-matrix $S(p_1,p_2)$ to be determined.

Plugging the ansatz (4.10) into the difference equation (4.9) gives the energy

$$E = 4\varepsilon_0 \sum_{j=1}^{2} \sin^2\left(\frac{p_j}{2}\right). \tag{4.11}$$

and equation (4.8) then determines the S-matrix:

$$S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}. \tag{4.12}$$

We note that $S(p_1,p_2) = S^{-1}(p_2,p_1)$. If p_1 and p_2 are both real, we also get $|S|^2 = 1$ implying probability conservation in scattering events. As in the one magnon sector, the momenta get fixed by the periodic boundary conditions, so imposing $\psi_{p_1p_2}(x_1,x_2) = \psi_{p_1p_2}(x_2,x_1+L)$, gives the equations

$$e^{ip_1L} = S(p_1,p_2), \qquad e^{ip_2L} = S(p_2,p_1). \tag{4.13}$$

This is the analog of the simple quantization condition in the one-magnon sector, but now there is also the possibility of complex solutions. A complex value of a momentum variable results in a decaying wavefunction, which means that the magnons form a bound state. Alternatively, we can view (4.13) as the result of translating one magnon J units along the spin chain. The acquired phase factor e^{ip_iJ} should be the result of scattering with the other magnon and is thus given by $S(p_i, p_j)$.

To solve the M-magnon problem, we will generalize (4.10) to a state labeled by M momentum variables consisting of terms that represent all possible scattering events. When inserting this ansatz into the Schrödinger equation, we always get an equation of the form (4.9) and the energy simply becomes a sum of M terms like those appearing in (4.11). We also get equations corresponding to (4.8) and these will determine the S-matrix.

In general, one would expect an M-body S-matrix to appear in these equations, but the fact that the hamiltonian is integrable implies that a general M-magnon scattering event factorizes into a number of two-magnon scattering events and the M-body S-matrix can be written as a product of the two-body S-matrices given by (4.12) [30]. The boundary conditions then give the momenta in terms of the S-matrix and we have the M Bethe equations

$$e^{ip_jJ} = \prod_{k\neq j}^{M} S(p_j, p_k). \tag{4.14}$$

Again, these equations simply express that translating one magnon by J units along the spin chain, gives a phase that is the product of phase factors resulting from scattering events with the other magnons.

It will be useful to work with the Bethe roots u_j, which are related to the momenta by

$$u_j = \frac{1}{2}\cot\frac{p_j}{2}. \tag{4.15}$$

Expressing the Bethe equations (4.14) in terms of these yields

$$e^{ip_jJ} = \left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{4.16}$$

In this expression and the ones to follow, $\sum_{k\neq j}^{M}$ simply means that k should be summed over all integers between 1 and M except j. In general there will be many sets $\{u_j\}$ satisfying these equations, and the Bethe roots comprising these sets are complex numbers. This means that there is a whole spectrum of energies corresponding to the M-magnon sector, depending on the distribution of Bethe roots. We introduce the Bethe states $|M, J; \{u_j\}\rangle$ which satisfy

$$H_{Sc}|M, J; \{u_j\}\rangle = E(M, J; \{u_j\})|M, J; \{u_j\}\rangle, \tag{4.17}$$

and the energy can then be expressed in terms of Bethe roots:

$$E(M,J;\{u_j\}) = 4\varepsilon_0 \sum_{j=1}^{M} \sin^2\left(\frac{p_j}{2}\right) = \varepsilon_0 \sum_{j=1}^{M} \frac{1}{u_j^2 + 1/4}. \tag{4.18}$$

Since we are about to identify these spin chains with single-trace operators of $\mathcal{N} = 4$ SYM, we also need to impose cyclicity on the states under consideration. This is distinct from the periodic boundary conditions that was already used to derive the Bethe equations. In the one magnon sector for example, the periodic boundary conditions simply meant that we identified the *sites* j and $j+J$, whereas cyclicity would mean that we should identify the *states* $|j\rangle$ and $|j+1\rangle$. More generally, we identify $|\downarrow \ldots\rangle$ with $|\ldots \downarrow\rangle$ and $|\uparrow \ldots\rangle$ with $|\ldots \uparrow\rangle$, where the dots represent a sequence of spins that are the same in the two corresponding states. The energy of the one magnon state $|1,J;p\rangle$ is now zero since $P_{j,j+1}|j\rangle_{cyclic} = |j+1\rangle_{cyclic} = |j\rangle_{cyclic}$. The cyclicity puts one more constraint on the momenta and thus the Bethe roots. Since the Bethe states carry definite momenta, we can define the translation operator on a M-magnon state as the product of M exponentials carrying the M momentum operators. Cyclic states are invariant under translation, so acting with the translation operator gives

$$T|M,J;\{u_j\}\rangle = e^{\sum_{j=1}^{M} ip_j}|M,J;\{u_j\}\rangle = |M,J;\{u_j\}\rangle \quad \Rightarrow \quad \prod_{j=1}^{M} e^{p_j} = 1, \tag{4.19}$$

or in terms of Bethe roots:

$$\prod_{j=1}^{M} \frac{u_j + i/2}{u_j - i/2} = 1. \tag{4.20}$$

4.2 SU(2) Spin Chains

We showed in the previous section (3.82) that the planar part of the 1-loop dilatation operator can be written

$$\mathcal{D}_1^{Planar} = \frac{\lambda}{8\pi^2} \sum_{i=1}^{J} (1 - P_{i,i+1}), \tag{4.21}$$

acting on the set of states

$$\mathcal{O}_{J,M} = Tr[XZZZXZZXZZ...], \tag{4.22}$$

with M factors of X and $J-M$ factors of Z appearing in the trace. The operators (4.21) and (4.2) are exactly the same if we set $\varepsilon_0 = \frac{\lambda}{8\pi^2}$. On the other hand, the states (4.22) can be mapped to cyclic spin chains by a simple change of notation. This means that we can associate

$$\mathcal{D}_1^{Planar} \leftrightarrow H_{Sc} \qquad with \qquad \varepsilon_0 = \frac{\lambda}{8\pi^2}, \tag{4.23}$$

$$Tr[XZZZXZZXZZ...] \leftrightarrow |\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow ...\rangle_{cyclic}. \tag{4.24}$$

A Bethe state $|M, J; \{u_j\}\rangle$ will now correspond to a linear combination of operators like (4.22) with one-loop anomalous dimension given by

$$\Delta_1(M, J; \{u_j\}) = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + 1/4}. \tag{4.25}$$

Solutions of the M nonlinear equations (4.16) that satisfy (4.20) are very hard to find for general M and J. One has to use numerical methods or consider certain limits in which the equations simplify.

4.2.1 Thermodynamic Limit

We will now consider the limit of $J, M \gg 1$. It is self-consistent to assume that the momenta scale as $1/J$, since it ensures that solutions to (4.16) will not be unstable to small variations in J. If we then set $p_j = k_j/J$, we get from (4.15)

$$u_j = \frac{i}{2}\left(\frac{1 + i\frac{k_j}{2J} + 1 - i\frac{k_j}{2J}}{1 + i\frac{k_j}{2J} - 1 + i\frac{k_j}{2J}}\right) = \frac{J}{k_j}, \tag{4.26}$$

and we see that u_j scales as J. We can then neglect the $1/4$ appearing in the denominator of (4.25) and write the anomalous dimension

$$\Delta_1(M, J; \{u_j\}) = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2}. \tag{4.27}$$

Taking the logarithm of (4.16) and expanding the arguments gives

$$\frac{J}{u_j} + 2\pi n_j = 2 \sum_{k \neq j}^{M} \frac{1}{u_j - u_k}, \tag{4.28}$$

where we used that $\log \frac{1+x}{1-x} \approx 2x$ for $x \ll 1$. The same procedure is applied to the momentum constraint (4.20) and we get

$$\sum_{j=1}^{M} \frac{1}{u_j} = 2\pi m. \tag{4.29}$$

The integers m and n_j appear in the equations, because we are taking the logarithm of complex numbers.[12] The solutions to the Bethe equations are thus characterized by the M integers n_j, and equation (4.29) gives an additional constraint that should be imposed on these solutions.

4.2.2 Rational Solution

We will now look for solutions with $n_j = n$ for all j. Multiplying (4.28) by $\frac{1}{u_j}$ and summing over j give

$$\sum_{j=1}^{M} \frac{J}{u_j^2} + 2\pi n \sum_{j=1}^{M} \frac{1}{u_j} = 2 \sum_{j=1}^{M} \frac{1}{u_j} \sum_{k \neq j}^{M} \frac{1}{u_j - u_k}. \tag{4.30}$$

The last term can be rewritten

$$\begin{aligned} \sum_{j=1}^{M} \frac{1}{u_j} \sum_{k \neq j}^{M} \frac{1}{u_j - u_k} &= \sum_{j=1}^{M} \sum_{k \neq j}^{M} \frac{1}{u_k} \Big(\frac{1}{u_j - u_k} - \frac{1}{u_j} \Big) \\ &= \sum_{j=1}^{M} \sum_{k \neq j}^{M} \frac{1}{u_k} \frac{1}{u_j - u_k} - \sum_{j=1}^{M} \sum_{k=1}^{M} \frac{1}{u_k u_j} + \sum_{j=1}^{M} \frac{1}{u_j^2}, \end{aligned}$$

which implies

$$2 \sum_{j=1}^{M} \frac{1}{u_j} \sum_{k \neq j}^{M} \frac{1}{u_j - u_k} = \sum_{j=1}^{M} \frac{1}{u_j^2} - \Big(\sum_{j=1}^{M} \frac{1}{u_j} \Big)^2. \tag{4.31}$$

Summing over j in (4.28) yields

$$\sum_{j=1}^{M} \frac{1}{u_j} = -2\pi n \frac{M}{J}, \tag{4.32}$$

where it was noted that the double sum vanishes due to symmetry in the indices. We note that the anomalous dimension is proportional to the first term on the left hand side in (4.30), so when inserting (4.31) and (4.32) we get

$$\Delta_1(M, J, n) = \lambda n^2 \frac{M}{2J^2} \Big(1 - \frac{M}{J} \Big), \tag{4.33}$$

where we neglected the sum over $1/u_j^2$ in (4.31), since it is already present with a factor of J in (4.30). We would like to express this result in a way that contains the number of Z's and X's on equal footing. After all, there is nothing special about choosing Z as a

[12]The complex number $w = e^z$ will not change if we let $z \to z + i2\pi n$ and therefore $\log w$ is only defined up to $i2\pi n$, where n is any integer.

background variable and we might as well have used X instead. The R-charges carried by the operators are $J_1 = M$ and $J_3 = J - M$, and we will introduce two corresponding integers m_1 and m_3. If we look at (4.29) and (4.32), we see that the momentum constraint can be written

$$nM + mJ = (n+m)J_1 + mJ_3 = 0. \tag{4.34}$$

To get an expression that is symmetric in the R-charges, we define

$$m_1 \equiv n + m, \quad m_3 \equiv m, \tag{4.35}$$

with which (4.34) can be written

$$m_1 J_1 + m_3 J_3 = 0. \tag{4.36}$$

We then use (4.34) to express (4.33) in terms of these new integers

$$\begin{aligned}
\Delta_1(J_1, J_3; m_1, m_3) &= \frac{\lambda}{2J^2}\left[n^2 M + nmM\right] \\
&= \frac{\lambda}{2J^2}\left[(m_1^2 - 2m_1 m_3 + m_3^2)J_1 + m_3(m_1 - m_3)J_1\right] \\
&= \frac{\lambda}{2J^2}\left[(m_1^2 - m_1 m_3)J_1 + m_3^2 J_3 - m_3^2 J_3\right] \\
&= \frac{\lambda}{2J^2}\left[m_1^2 J_1 + m_3^2 J_3 - (m_1 J_1 + m_3 J_3)m_3\right] \\
&= \frac{\lambda}{2J^2}\left(m_1^2 J_1 + m_3^2 J_3\right),
\end{aligned} \tag{4.37}$$

where equation (4.36) was used in the last line. This expression appears to depend on one more integer than (4.33), but this expression should be supplemented by (4.36), which could be used to eliminate one of the variables above. However, we prefer to keep the expression in the symmetric form above with the additional constraint (4.36).

4.3 SU(3) Spin Chains

The Bethe ansatz can be extended to more general cases than the $SU(2)$ spin chain considered above. These will also be referred to as spin chains although they do not have a direct analog in magnetism. The case of interest here is the $SU(3)$ cyclic spin chains, which correspond to the generic operators

$$\mathcal{O}_{J_1 J_2 J_3} = Tr[ZZXZZYZZXZYYZ\ldots], \tag{4.38}$$

carrying J_1 factors of X, J_2 factors of Y, and J_3 factors of Z. At tree-level, the operators all have conformal dimension $\Delta = J = J_1 + J_2 + J_3$, but as we have seen, they mix at one-loop, and we will have to diagonalize the dilatation operator in order to obtain the anomalous dimension. We can still write the planar part of the dilatation operator to one-loop

$$\mathcal{D}_1^{Planar} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{J}(1 - P_{i,i+1}), \tag{4.39}$$

but if we were to write it in terms of matrices, we would have to use the eight Gell-Mann matrices instead of the three Pauli matrices used in (4.2). Each site in the operators (4.38) carries the fundamental representation of $SU(3)$.

In the case of $SU(2)$ spin chains, integrability ensured that the solutions could be written down in terms of the Bethe equations. Thus, the first question we should address is whether the $SU(3)$ spin chain is integrable. Instead of constructing an R-matrix along the lines of [29], we will show that an appropriately defined S-matrix satisfies a certain Yang-Baxter relation. This relation reflects that a multi-particle scattering event factorizes into a product of two-particle scattering events, which implies that the spin chain is integrable.

4.3.1 The S-matrix

As in the case of $SU(2)$ operators, we consider excitations of the reference vacuum state

$$|0\rangle_J = Tr[Z^J], \tag{4.40}$$

which do not receive radiative corrections and has conformal dimension $\Delta = J$.

Compared to the $SU(2)$ sector, the situation at hand is complicated by the fact that there is now a possibility of the two different types of excitations X and Y. Excitations involving both complex scalars can mix, and the S-matrix is no longer just a function, but indeed a matrix. For example, the two magnon sector now contains four classes of states: $|XX\rangle$, $|XY\rangle$, $|YX\rangle$, and $|YY\rangle$. Each of these can be written as the superposition

$$|\xi\eta\rangle = \sum_{0\le x_1<x_2\le J} \psi_{\xi\eta}(x_1,x_2)|ZZ\ldots\xi_{x_1}ZZZ\ldots\eta_{x_2}ZZ\ldots\rangle, \tag{4.41}$$

where ξ,η can label X or Y. The states $|XX\rangle$ and $|YY\rangle$ can be analyzed as in the $SU(2)$ sector using a coordinate Bethe ansatz reflecting the possibility of scattering:

$$\psi_{XX}(x_1,x_2) = \psi_{YY}(x_1,x_2) = e^{ip_1x_1+p_2x_2} + s(p_2,p_1)e^{ip_2x_1+p_1x_2}, \tag{4.42}$$

with $s(p_1,p_2) = S_{SU(2)}(p_1,p_2)$. The states $|XY\rangle$ and $|YX\rangle$ are a bit more tricky because of mixing. If X and Y are nearest neighbors in such a state, the permutation operator will return both kinds of states. Again, we associate two momenta with such states and we can represent them schematically by

$$|XY;p_1,p_2\rangle : \qquad A_{in}\Big(X \longrightarrow\leftarrow Y\Big) + A_{out}\Big(X \longleftarrow\longrightarrow Y\Big), \tag{4.43}$$

$$|YX;p_1,p_2\rangle : \qquad B_{in}\Big(Y \longrightarrow\leftarrow X\Big) + B_{out}\Big(Y \longleftarrow\longrightarrow X\Big), \tag{4.44}$$

where the arrows represent the two momenta. Again, the idea is that the magnons can scatter which results in exchange of momenta. If the system is integrable, the two momenta should be separately conserved in the process. The notation implies that we can view the wavefunction as a superposition of an incoming and an outgoing wave. The choice of in and out states are rather arbitrary, but it is a useful picture to have in mind when we consider it as a scattering process.

The coordinate Bethe ansatz for these states is then

$$\psi_{XY}(x_1, x_2) = A_{in} e^{ip_1 x_1 + ip_2 x_2} + A_{out} e^{ip_2 x_1 + ip_1 x_2}, \tag{4.45}$$

$$\psi_{YX}(x_1, x_2) = B_{in} e^{ip_1 x_1 + ip_2 x_2} + B_{out} e^{ip_2 x_1 + ip_1 x_2}. \tag{4.46}$$

It is important that we have two constants in each of these ansatze instead of just one as we had in the $SU(2)$ sector. The complete wavefunction also includes the part in (4.42) and we cannot include A_{in} or B_{in} in the normalization.

The two states (4.45) and (4.46) mix under scattering. For instance, considering the in state $X \longrightarrow\leftarrow Y$, Two things can happen: The magnons are reflected resulting in the second term of $|XY\rangle$ or the magnons pass each other resulting in the second term of $|YX\rangle$. Conversely, the second term of $|YX\rangle$ can result from either of two events: transmission of an $|XY\rangle$ state or reflection of a $|YX\rangle$ state. If we let $t(p_2, p_1)$ denote the amplitude of transmission and $r(p_2, p_1)$ denote the amplitude of reflection, this can be summarized in matrix notation (choosing the transmission diagonal representation)

$$\mathbf{c}_{out} \equiv \begin{pmatrix} B_{out} \\ A_{out} \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} A_{in} \\ B_{in} \end{pmatrix} \equiv \begin{pmatrix} t & r \\ r & t \end{pmatrix} \mathbf{c}_{in}. \tag{4.47}$$

Thus, the amplitude for a state initially in A_{in} scattering to B_{out} is given by t etc. In general one should allow four distinct entries in the S-matrix, but the symmetry between X and Y excitations makes it the matrix symmetric. Including the $|XX\rangle$ and $|YY\rangle$ states in this notation, we can represent the complete scattering process by the 4×4 S-matrix

$$S(p_1, p_2) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & r & 0 \\ 0 & r & t & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \tag{4.48}$$

and a scattering event can thus be written $\mathbf{C}_{out} = S(p_1, p_2)\mathbf{C}_{in}$, with $\mathbf{C}_{in} = (1, A_{in}, B_{in}, 1)$.

We note that the S-matrix has to be unitary due to probability conservation:

$$|\mathbf{C}_{in}|^2 = |\mathbf{C}_{out}|^2 = \mathbf{C}_{in}^* S^\dagger S \mathbf{C}_{in} \Rightarrow S^\dagger S - 1 = 0, \tag{4.49}$$

and this implies that

$$|t|^2 + |r|^2 = 1, \qquad tr^* + t^* r = 0, \tag{4.50}$$

since we already saw that $|s^2| = 1$ for real momenta.

To determine the reflection and transmission coefficients b and c, we focus on the mixed states which are eigenvalues of the momentum operator. The most general state of this kind will be given by the superposition

$$|\{XY\}, J\rangle = \sum_{0\le x_1<x_2\le J} \psi_{XY}(x_1,x_2)|x_1,x_2\rangle_{XY} + \psi_{YX}(x_1,x_2)|x_1,x_2\rangle_{YX}. \tag{4.51}$$

This state is inserted into the eigenvalue equation for the one-loop dilatation operator. We note that states with $x_2 > x_1 + 1$ will not mix and collecting coefficients proportional to states with a given x_1 and x_2, we get

$$\begin{aligned}\Delta_1\psi_{XY}(x_1,x_2) =& 2\psi_{XY}(x_1,x_2) - \psi_{XY}(x_1-1,x_2) - \psi_{XY}(x_1+1,x_2) \\ &+ 2\psi_{XY}(x_1,x_2) - \psi_{XY}(x_1,x_2-1) - \psi_{XY}(x_1,x_2+1),\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}\Delta_1\psi_{YX}(x_1,x_2) =& 2\psi_{YX}(x_1,x_2) - \psi_{YX}(x_1-1,x_2) - \psi_{YX}(x_1+1,x_2) \\ &+ 2\psi_{YX}(x_1,x_2) - \psi_{YX}(x_1,x_2-1) - \psi_{YX}(x_1,x_2+1).\end{aligned} \tag{4.53}$$

Inserting the ansatz (4.45) or (4.46) into either of these equations gives the anomalous dimension in terms of the momenta. The result is exactly the same as we got in the $SU(2)$ sector (we neglect the factor in front of the dilatation operator for now):

$$\tilde{\Delta}_1(p_1,p_2) = 4 - e^{ip_1} - e^{-ip_1} - e^{ip_2} - e^{-ip_2}. \tag{4.54}$$

The S-matrix is determined from the coefficients of the "contact terms" with $x_2 = x_1 + 1$. Collecting these terms after the generic state above is inserted, gives the coupled equations

$$\begin{aligned}\tilde{\Delta}_1\psi_{XY}(x_1,x_2) =& 3\psi_{XY}(x_1,x_2) - \psi_{YX}(x_1,x_2) \\ &- \psi_{XY}(x_1-1,x_2) - \psi_{XY}(x_1,x_2+1),\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}\tilde{\Delta}_1\psi_{YX}(x_1,x_2) =& 3\psi_{YX}(x_1,x_2) - \psi_{XY}(x_1,x_2) \\ &- \psi_{YX}(x_1-1,x_2) - \psi_{YX}(x_1,x_2+1).\end{aligned} \tag{4.56}$$

The Bethe ansatz (4.45) and (4.46) is now inserted into these equations and we get two algebraic equations that relate A_{out} and B_{out} to A_{in} and B_{in}:

$$\begin{aligned}0 &= (1 + e^{ip_1+ip_2} - e^{ip_2})A_{in} + (1 + e^{ip_1+ip_2} - e^{ip_1})A_{out} - e^{ip_2}B_{in} - e^{ip_1}B_{out}, \\ 0 &= (1 + e^{ip_1+ip_2} - e^{ip_2})B_{in} + (1 + e^{ip_1+ip_2} - e^{ip_1})B_{out} - e^{ip_2}A_{in} - e^{ip_1}A_{out}.\end{aligned}$$

t and r can be found from these equations by comparison to the defining equation (4.47). The elements of the S-matrix are thus

$$s(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}, \tag{4.57}$$

$$t(p_1,p_2) = \frac{e^{ip_1} - e^{ip_2}}{1 + e^{ip_1+ip_2} - 2e^{ip_2}}, \tag{4.58}$$

$$r(p_1,p_2) = \frac{e^{ip_1} + e^{ip_2} - e^{ip_1+ip_2} - 1}{1 + e^{ip_1+ip_2} - 2e^{ip_2}}, \tag{4.59}$$

and one can verify that these satisfy the unitarity condition (4.50).

4.3.2 Integrability and the Yang-Baxter Equation

A necessary condition for integrability is that the S-matrix satisfies the Yang-Baxter equation, which reads

$$S_{1,2}(p_1,p_2)S_{1,3}(p_1,p_3)S_{2,3}(p_1,p_2) = S_{2,3}(p_2,p_3)S_{1,3}(p_1,p_3)S_{1,2}(p_1,p_2). \tag{4.60}$$

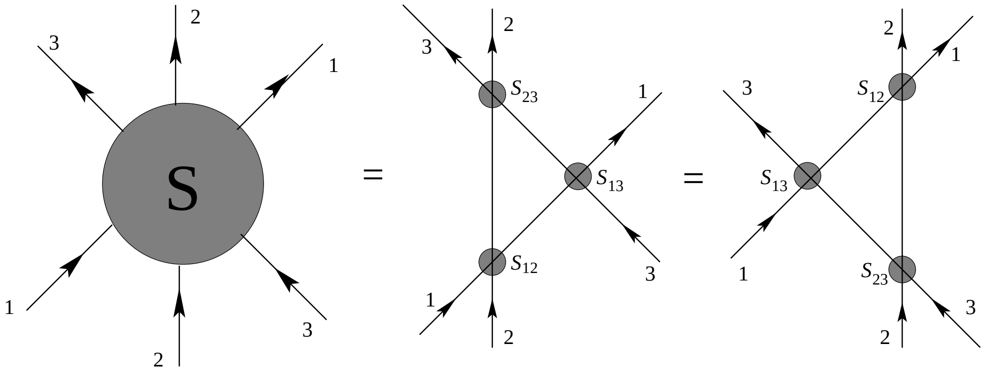

Figure 7: Diagrammatic representation of the Yang-Baxter equation: The S-matrix of a three-particle scattering event factorizes into three two-particle scattering events, and the Yang-Baxter equation says that the order of these can be intertwined as indicated by these diagrams. The three momenta should be separately conserved in the process.

This relation is depicted in figure 7 and reflects that multi-particle scattering events factorize into a number of two-particle scattering events. Since three magnons are involved in this relation, each matrix acts in an eight-dimensional Hilbert space. The matrices act such that the particle not involved is left invariant by the action and the four-dimensional subspace spanned by the two participating particles is acted on by the matrix in (4.48). We define the ordered basis of the eight states $\{|i\rangle \otimes |j\rangle \otimes |k\rangle\}$ by

$$\{|XXX\rangle, |XXY\rangle, |XYX\rangle, |XYY\rangle, |YXX\rangle, |YXY\rangle, |YYX\rangle, |YYY\rangle\}, \tag{4.61}$$

and the matrices are then given by

$$S_{1,2} = \begin{pmatrix} s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & r & 0 & 0 \\ 0 & 0 & r & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{pmatrix}, \qquad S_{1,3} = \begin{pmatrix} s' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t' & 0 & 0 & r' & 0 & 0 & 0 \\ 0 & 0 & s' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t' & 0 & 0 & r' & 0 \\ 0 & r' & 0 & 0 & t' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s' & 0 & 0 \\ 0 & 0 & 0 & r' & 0 & 0 & t' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s' \end{pmatrix},$$

$$S_{2,3} = \begin{pmatrix} s'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t'' & r'' & 0 & 0 & 0 & 0 & 0 \\ 0 & r'' & t'' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s'' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t'' & r'' & 0 \\ 0 & 0 & 0 & 0 & 0 & r'' & t'' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s'' \end{pmatrix}, \tag{4.62}$$

where $s = s(p_1, p_2)$, $s' = s(p_1, p_3)$, $s'' = s(p_2, p_3)$ and so on. Inserting these matrices into the Yang-baxter equation results in the three equations

$$\begin{aligned} sr's'' - rs'r'' - tr't'' &= 0, \\ ts'r'' - st'r'' + rr't'' &= 0, \\ rt's'' - tr'r'' - rs't'' &= 0. \end{aligned} \tag{4.63}$$

These relations can easily be verified using the expressions (4.57)-(4.59) and we thus conclude that scattering factorizes and the system can be assumed to be integrable.[13]

4.3.3 Nested Bethe Ansatz

Integrability implies factorized and non-diffractive scattering, and the wavefunction of an M-magnon state should be some superposition of plane waves involving all permutations of the M momentum variables generalizing (4.42), (4.45), and (4.46). One can then verify that inserting such an ansatz in the schrödinger equation and considering a generic M-magnon version of (4.52) will yield the eigenvalues

$$\Delta_1(M_1, M_2, J; \{u_j\}) = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M_1} \frac{1}{u_j^2 + 1/4}, \tag{4.64}$$

as we had in the $SU(2)$ sector. The anomalous dimension is thus determined once we have a set of $SU(3)$ Bethe roots $\{u_j\}$.

We will now derive the $SU(3)$ Bethe equations that determine the roots u_j. These should be a generalization of equation (4.16) and they are derived in the same manner, although the details are a bit more complicated. The technique is called the nested Bethe ansatz, since one has to "go up a level" and consider new excitations on a spin chain of excitations.[14]

Denoting the total number of excitations M_1 and the number of X and Y excitations $M_1 - M_2$ and M_2 respectively, the R-charges can be written

$$(J_1, J_2, J_3) = (M_1 - M_2, M_2, J - M_1). \tag{4.65}$$

[13]The fact that the S-matrix satisfies the Yang-Baxter equation provides strong evidence that the system is integrable, but it is does not prove it.

[14]The term "nested Bethe ansatz" stems from the Chinese nest of boxes: A spin chain inside another spin chain.

Our starting point is the Bethe equations for the $SU(2)$ sector. These were derived by translating a magnon once around the spin chain and equating the total phase shift with the product of all phase factors resulting from scattering with the remaining magnons. With the present notation such a state corresponds to $M_2 = 0$ and the Bethe equations read

$$e^{ip_k J}|M_1, 0, \{p_j\}\rangle = s_{k,k+1} \ldots s_{k,M_1} s_{k,1} \ldots s_{k,k-1}|M_1, 0, \{p_j\}\rangle, \tag{4.66}$$

where $s_{i,j} = s(p_i, p_j)$. In this case, the states on both sides of the equation are clearly redundant, since the prefactors are just functions, but when we generalize this to $M_2 \neq 0$ the expression becomes a matrix equation that we have to diagonalize. It will be convenient to work with the "short" spin chain $|\Psi\rangle$ obtained by omitting all the background Z's from a given spin chain. The short spin chain thus has length M_1 with M_2 Y's, which will be referred to as excitations of the short spin chain. The $SU(2)$ state in (4.66) can be regarded as the ground state of the short spin chain. With a general short spin chain $|\Psi\rangle$, the matrix Bethe equations become

$$e^{ip_k J}|\Psi\rangle = S_{k,k+1} \ldots S_{k,M_1} S_{k,1} \ldots S_{k,k-1}|\Psi\rangle, \tag{4.67}$$

where $S_{i,j} = S_{i,j}(p_i, p_j)$ is the S-matrix given in (4.48) acting on sites i, j in the short spin chain. Defining the reduced S-matrix

$$\tilde{S}_{i,j}(p_i, p_j) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{t}_{i,j} & \tilde{r}_{i,j} & 0 \\ 0 & \tilde{r}_{i,j} & \tilde{t}_{i,j} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.68}$$

where $\tilde{t}_{i,j} \equiv t_{i,j}/s_{i,j}$, $\tilde{r}_{i,j} \equiv r_{i,j}/s_{i,j}$, the matrix Bethe equations can be written

$$\lambda_k|\Psi\rangle = \tilde{S}_{k,k+1} \ldots \tilde{S}_{k,M_1} \tilde{S}_{k,1} \ldots \tilde{S}_{k,k-1}|\Psi\rangle, \tag{4.69}$$

where we have defined the eigenvalue λ_k by

$$\lambda_k = e^{ip_k J} s_{k+1,k} \ldots s_{M_1,k} s_{1,k} \ldots s_{k-1,k}, \tag{4.70}$$

and used that $s_{i,j}^{-1} = s_{j,i}$.

We will now solve the one-magnon problem of the short spin chain. We will again introduce a coordinate space wavefunction $\psi(x) = \psi_x$ and write the one-magnon state as

$$|\Psi\rangle = \sum_{x=1}^{M_1} \psi_x |x\rangle, \tag{4.71}$$

where $|x\rangle$ is a state with a Y inserted at position x. This has to be inserted into (4.69) to obtain expressions for ψ_x and λ_k. Note that a simple Fourier transform will not solve the problem, since the short spin chain is not homogeneous. Instead, we can obtain a recursion relation for ψ_x by acting on $|\Psi\rangle$ with the S-matrices appearing in (4.69) one by one and collecting terms along the way. For example, starting with $\tilde{S}_{k,k-1}$ gives

$$\tilde{S}_{k,k-1}|\Psi\rangle = \Big(\psi_{k-1}\tilde{t}_{k,k-1} + \psi_k\tilde{r}_{k,k-1}\Big)|k-1\rangle + \Big(\psi_k\tilde{t}_{k,k-1} + \psi_{k-1}\tilde{r}_{k,k-1}\Big)|k\rangle + \sum_{x\neq\{k,k-1\}}^{M_1} \psi_x|x\rangle. \tag{4.72}$$

None of the remaining matrices in (4.69) will change the state $|k-1\rangle$ or produce new terms proportional to it, so at this stage, we can equate the prefactor of this state with $\lambda_k\psi_k$, which is the associated factor on the left hand side of (4.69). Similarly, acting on the right hand side of (4.72) with $\tilde{S}_{k,k-2}$ produces states proportional to $|k-2\rangle$ and this is the only contribution to such states from the whole string of matrices in (4.69). Proceeding like this, we get expressions relating the wavefunction at different points:

$$\lambda_k\psi_{k-1} = \tilde{r}_{k,k-1}\psi_k + \tilde{t}_{k,k-1}\psi_{k-1}, \tag{4.73}$$

$$\lambda_k\psi_{k-2} = \tilde{r}_{k,k-2}\big(\tilde{t}_{k,k-1}\psi_k + \tilde{r}_{k,k-1}\psi_{k-1}\big) + \tilde{t}_{k,k-2}\psi_{k-2}, \tag{4.74}$$

$$\lambda_k\psi_{k-3} = \tilde{r}_{k,k-3}\big(\tilde{t}_{k,k-2}\big[\tilde{t}_{k,k-1}\psi_k + \tilde{r}_{k,k-1}\psi_{k-1}\big] + \tilde{r}_{k,k-2}\psi_{k-2}\big) + \tilde{t}_{k,k-3}\psi_{k-3}, \tag{4.75}$$

and so on. These equations allow one to derive a general expression for the wavefunction. We start by relating ψ_k and ψ_{k-1}:

$$\frac{\psi_{k-1}}{\psi_k} = \frac{\tilde{r}_{k,k-1}}{\lambda_k - \tilde{t}_{k,k-1}}. \tag{4.76}$$

Using that $\tilde{t}_{k,k-1}\psi_k = \frac{\tilde{t}_{k,k-1}}{\tilde{r}_{k,k-1}}\psi_{k-1}(\lambda_k - \tilde{t}_{k,k-1})$, we can derive an expression relating ψ_{k-2} and ψ_{k-1} by means of (4.74):

$$\frac{\psi_{k-2}}{\psi_{k-1}} = \frac{\tilde{r}^2_{k,k-1} - \tilde{t}^2_{k,k-1} + \lambda_k\tilde{t}_{k,k-1}}{\lambda_k - \tilde{t}_{k,k-2}}\Big(\frac{\tilde{r}_{k,k-2}}{\tilde{r}_{k,k-1}}\Big). \tag{4.77}$$

This expression can now be extended to a general relation between ψ_{k-j-1} and ψ_{k-j} as we will see when going to the next level. The reason is that we need to determine the expression $\tilde{t}_{k,k-2}\big[\tilde{t}_{k,k-1}\psi_k + \tilde{r}_{k,k-1}\psi_{k-1}\big]$ appearing in (4.75) in terms of ψ_{k-2}, but this is done using (4.74), and the result can be obtained from $\tilde{t}_{k,k-1}\psi_k$ as stated above, by letting $k-1 \to k-2$. We then immediately see that the proportionality factor between ψ_{k-3} and ψ_{k-2} can be obtained from (4.77) by substituting $k-1$ with $k-2$ and $k-2$ with $k-3$ on the right hand side. In fact, since the equations obtained from $|k-j\rangle$ and $|k-j-1\rangle$ are related in the same manner as (4.74) and (4.75), the argument holds true for any j, and we thus get the recursion relation

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{\tilde{r}^2_{k,k-j} - \tilde{t}^2_{k,k-j} + \lambda_k\tilde{t}_{k,k-j}}{\lambda_k - \tilde{t}_{k,k-j-1}}\Big(\frac{\tilde{r}_{k,k-j-1}}{\tilde{r}_{k,k-j}}\Big). \tag{4.78}$$

We now reintroduce the Bethe roots u_j associated with the momentum variables p_j. They are defined through

$$e^{ip_j} = \frac{u_j + i/2}{u_j - i/2}. \tag{4.79}$$

In terms of these s, t, and r given in (4.57)-(4.59) become

$$s_{i,j} = \frac{u_i - u_j + i}{u_i - u_j - i}, \qquad t_{i,j} = \frac{u_i - u_j}{u_i - u_j - i}, \qquad r_{i,j} = \frac{i}{u_i - u_j - i} \tag{4.80}$$

$$\tilde{t}_{i,j} = \frac{u_i - u_j}{u_i - u_j + i}, \qquad \tilde{r}_{i,j} = \frac{i}{u_i - u_j + i}. \tag{4.81}$$

Using that $\tilde{r}_{i,j}^2 - \tilde{t}_{i,j}^2 = -\frac{u_i - u_j - i}{u_i - u_j + i}$, we can express the recursion relation in terms of the Bethe roots and get

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{u_k - \frac{i}{1-\lambda_k} - u_{k-j}}{u_k - \frac{i\lambda_k}{1-\lambda_k} - u_{k-j-1}} = \frac{u_k - \frac{i}{2}\left(\frac{1+\lambda_k}{1-\lambda_k}\right) - i/2 - u_{k-j}}{u_k - \frac{i}{2}\left(\frac{1+\lambda_k}{1-\lambda_k}\right) + i/2 - u_{k-j-1}}. \tag{4.82}$$

Now, we observe that the left hand side only depends on $k - j$. Thus, the right hand side should remain constant if we change k while keeping $k - j$ fixed. This is only consistent if

$$q \equiv u_k - \frac{i}{2}\Big(\frac{1 + \lambda_k}{1 - \lambda_k}\Big), \tag{4.83}$$

is a constant. The recursion relation then becomes a lot simpler and one can easily verify that

$$\psi_x = \psi_x(q) = \prod_{j=1}^{x-1} \frac{u_j - q - i/2}{u_{j+1} - q + i/2}. \tag{4.84}$$

solves (4.82). We will also need an expression for λ_k, which is obtained by inverting (4.83):

$$\lambda_k = \lambda_k(q) = \frac{u_k - q - i/2}{u_k - q + i/2}. \tag{4.85}$$

We have made the dependencies of q explicit in the expression for λ_k and ψ_x, since we may interpret this constant as a Bethe root parameterizing the momentum of the Y excitation in the short chain. This root does not substitute any of the M_1 Bethe roots u_j, since these are still parameterizing the M_1 momenta associated with the Y and $M_1 - 1$ X excitations in the original chain. It is an auxiliary root that should be determined by the boundary conditions. Imposing periodic boundary conditions on the short chain yields

$$\prod_{k=1}^{M_1} \lambda_k = 1, \tag{4.86}$$

which quantizes q. Expressing this equation in terms of Bethe roots and using the definition of λ_k (4.70), we get the following $M_1 + 1$ equations for the roots

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \frac{u_j - q - i/2}{u_j - q + i/2} \prod_{k \neq j}^{M_1} \frac{u_j - u_k + i}{u_j - u_k - i}, \tag{4.87}$$

$$1 = \prod_{k=1}^{M_1} \frac{u_k - q - i/2}{u_k - q + i/2}, \tag{4.88}$$

and this completely solves the one-magnon sector of the short spin chain.

It is useful to pause and compare these findings with the one-magnon results for the original "long" spin chain. The boundary conditions there gave us the simple quantization condition on the momentum: $e^{ipJ} = 1$. To see the correspondence with the short spin chain we define the shifted Bethe roots $\tilde{q}_k \equiv q - u_k$. We can then write $\lambda_k = e^{i\tilde{p}_k}$, where $\tilde{p}_k$ is the momentum associated with the shifted root $\tilde{u}_k$. Quantizing the momentum variable then looks similar in the two cases:

$$\text{Long spin chain:} \qquad \prod_{k=1}^{J} e^{ip} = 1, \tag{4.89}$$

$$\text{Short spin chain:} \qquad \prod_{k=1}^{M_1} e^{i\tilde{p}_k} = 1. \tag{4.90}$$

The first relation is just a restatement of the fact that translating one magnon through the complete spin chain should yield the identity, since the background is the translation invariant vacuum (string of Z's). We would like to use the same picture on the short spin chain, but we should be careful in which sense we regard the background of X's as a vacuum. Of course, the short chain is just a useful construction and the Y excitation scatters just as much on every X as it would on other Y excitations. The momentum corresponding to q parameterizes the Y excitation on the short chain, but as the short chain is not translation invariant, translating the magnon one time through the short spin chain does not yield the identity. Thus, we cannot write a simple quantization condition like $e^{ip'M_1} = 1$, where p' is the momentum associated with q. Nevertheless, we can think of all scattering with the X's as being encoded in the shifted momenta $\tilde{p}_k$: Translating the Y magnon through the short spin chain results in the identity if we use the M_1 shifted momenta $\tilde{p}_k$ instead of just p'. This is the content of equation (4.90), which we interpret as the imposition of periodic boundary conditions on an inhomogeneous spin chain.

With the above considerations in mind, we move on to the two-magnon state of the short spin chain. We now expect the appearance of two auxiliary roots q_1 and q_2 parameterizing the two magnons. The state is described by a two-particle wavefunction $\psi_{x_1,x_2}(q_1, q_2)$, and we can write a general two-magnon state as the superposition

$$|\Psi\rangle = \sum_{x_1 \leq x_2 \leq M_1} \psi_{x_1,x_2}(q_1, q_2)|x_1 x_2\rangle, \tag{4.91}$$

where x_1 and x_2 denote the position of the Y's. Since we have just argued that a magnon on the short spin chain in a certain sense propagates as if it was free, we will let us inspire by the two-magnon wavefunction ansatz used in the $SU(2)$ sector and make the following (secondary) Bethe ansatz:

$$\psi_{x_1,x_2}(q_1, q_2) = \psi_{x_1}(q_1)\psi_{x_2}(q_2) + S'\psi_{x_1}(q_2)\psi_{x_2}(q_1). \tag{4.92}$$

Once again, the idea is that the Y magnons should either propagate freely or exchange momenta with an amplitude S', but the situation is much more complicated than the true two-magnon sector of the $SU(2)$ spin chain, since the wavefunction depends on the M_1 Bethe roots u_k in addition to the two new auxiliary roots. We make the following ansatz for the two-magnon eigenvalue: $\lambda_k^{(2)}(q_1, q_2) = \lambda_k(q_1)\lambda_k(q_2)$ and insert this and (4.91) into the eigenvalue equation (4.69). One can then match terms on both side of the equation to get an expression for S' [31]:

$$S' = \frac{\lambda_k(q_2)\psi_k(q_2)\psi_{k-j}(q_1)\lambda_{k-1}(q_1)\ldots\lambda_{k-j}(q_1) + \psi_k(q_1)\psi_{k-j}(q_2)\lambda_k(q_2)\ldots\lambda_{k-j}(q_2)}{\lambda_k(q_1)\psi_k(q_1)\psi_{k-j}(q_2)\lambda_{k-1}(q_2)\ldots\lambda_{k-j}(q_2) + \psi_k(q_2)\psi_{k-j}(q_1)\lambda_k(q_1)\ldots\lambda_{k-j}(q_1)}. \tag{4.93}$$

Using (4.85) and (4.84) we get the amazingly simple result

$$S' = S'(q_1, q_2) = \frac{q_1 - q_2 + i}{q_1 - q_2 - i}, \tag{4.94}$$

which is just the $SU(2)$ S-matrix with the auxiliary momenta as arguments. A priori, there was no guarantee that S' would only depend on the two auxiliary roots, but indeed it does. In fact, we get the exact same result as if we had considered two excitations on a true vacuum state, indicating that we can take the above picture of free short chain magnons very literally. The boundary conditions on the wavefunction result in Bethe equations that determine the auxiliary roots, and these are now trivial generalizations of the $SU(2)$ conditions. The identity on the right hand side of (4.90) is simply replaced by S' giving

$$\prod_{k=1}^{M_1} e^{i\tilde{p}_k^{(1)}} = S'(q_1, q_2), \qquad \prod_{k=1}^{M_1} e^{i\tilde{p}_k^{(2)}} = S'(q_2, q_1), \tag{4.95}$$

where $\tilde{p}_k^{(1)}$ are the shifted roots corresponding to q_1 and $\tilde{p}_k^{(2)}$ are the shifted roots corresponding to q_2. These equations say that shifting one magnon once around the chain produces a phase factor that is the result of scattering with the other magnon. Since the short spin chain inherits factorized scattering from the long spin chain, the results are now readily generalized to the M_2-magnon sector of the short spin chain and therefore the general M_1-magnon problem of the $SU(3)$ spin chain. Translating a magnon once around the short chain produces a phase factor that is a product of $M_2 - 1$ scattering events. Hence, we get the complete set of Bethe equations

$$\prod_{k=1}^{M_2} \lambda_j(q_k) = e^{ip_j J} s_{j+1,j} \dots s_{M_1,j} s_{1,j} \dots s_{j-1,j}, \tag{4.96}$$

$$\prod_{k=1}^{M_1} e^{i\tilde{p}_k^{(j)}} = \prod_{k\neq j}^{M_2} S'(q_j, q_k), \tag{4.97}$$

or expressed in terms of the roots

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k\neq j}^{M_1} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{M_2} \frac{u_j - q_k - i/2}{u_j - q_k + i/2}, \tag{4.98}$$

$$1 = \prod_{k\neq j}^{M_2} \frac{q_j - q_k + i}{q_j - q_k - i} \prod_{k=1}^{M_1} \frac{q_j - u_k - i/2}{q_j - u_k + i/2}. \tag{4.99}$$

These $M_1 + M_2$ equations determine the M_1 Bethe roots u_j and the M_2 auxiliary roots q_j and thus completely solve the general M_1 magnon sector of the $SU(3)$ spin chain. In addition, we should also impose the cyclicity constraint which is derived as in the $SU(2)$ sector giving:

$$1 = \prod_{k=1}^{M_1} \frac{u_k + i/2}{u_k - i/2}. \tag{4.100}$$

4.3.4 Rational Solution

We will now follow [32] and show that a certain set of rational solutions can be obtained in the thermodynamic limit much the same way as we did with $SU(2)$ operators. We start by taking the logarithm of (4.98) and (4.99). This introduces $M_1 + M_2$ integers in the equations, one for each root just as in (4.28). To find a rational solution, we choose the M_1 integers associated with the u_j's to be equal $n_{u_j} = n$ and the M_2 integers associated with the q_j's to be equal $n_{q_j} = m$. We then consider the limit $J \gg 1$ and expand the arguments of the logarithms. The Bethe equations then become

$$\frac{J}{u_j} + 2\pi n = 2\sum_{k\neq j}^{M_1} \frac{1}{u_j - u_k} - \sum_{k=1}^{M_2} \frac{1}{u_j - q_k}, \tag{4.101}$$

$$2\pi m = 2\sum_{k\neq j}^{M_2} \frac{1}{q_j - q_k} - \sum_{k=1}^{M_1} \frac{1}{q_j - u_k}. \tag{4.102}$$

Taking the logarithm of both sides of the constraint (4.100) and expanding gives

$$2\pi p + \sum_{j=1}^{M_1} \frac{1}{u_j} = 0, \tag{4.103}$$

and the anomalous dimension becomes

$$\Delta_1(M_1, M_2, J; n, m) = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M_1} \frac{1}{u_j^2}. \tag{4.104}$$

We now solve the Bethe equations for $\sum_j 1/u_j^2$. The principle is the same as in the $SU(2)$ sector, but a few more steps are needed so we give the derivations in detail. In the following, we will suppress the limits on summations, since they are implicit in the variables to be summed. The sum runs from 1 to M_1 if the subscript is on a u and from 1 to M_2 if the subscript is on a q. We start by summing over j in (4.101) and (4.102). Combining the two gives

$$\sum_{j,k} \frac{1}{u_j - q_k} = 2\pi m M_2, \qquad J \sum_j \frac{1}{u_j} = -2\pi(nM_1 + mM_2). \tag{4.105}$$

We then multiply equation (4.101) by $\frac{1}{u_j}$ and sum over j. Using (4.31) and neglecting the $1/u_j^2$ sum that does not carry a factor of J, gives

$$J \sum_j \frac{1}{u_j^2} + 2\pi n \sum_j \frac{1}{u_j} = -\Big(\sum_j \frac{1}{u_j} \Big)^2 - \sum_{j,k} \frac{1}{u_j(u_j - q_k)}. \tag{4.106}$$

We see that $\sum_j 1/u_j^2$ and therefore the anomalous dimension can be determined from (4.105) and (4.106) if we can calculate $\sum_{j,k} \frac{1}{u_j(u_j-q_k)}$. To do, this we start by noting the relations

$$2 \sum_{l,j,k\neq j} \frac{1}{(u_j - u_k)(q_l - u_j)} = \sum_{l,j,k} \frac{1}{(q_l - u_k)(q_l - u_j)} - \sum_{l,k} \Big(\frac{1}{q_l - u_k} \Big)^2, \tag{4.107}$$

$$2 \sum_{l,j,k\neq j} \frac{1}{(q_j - q_k)(u_l - q_j)} = \sum_{l,j,k} \frac{1}{(u_l - q_k)(u_l - q_j)} - \sum_{l,k} \Big(\frac{1}{q_l - u_k} \Big)^2. \tag{4.108}$$

We then multiply (4.101) by $\sum_l \frac{1}{q_l - u_j}$, multiply (4.102) by $\sum_l \frac{1}{u_l - q_j}$, sum over j in both equations, and finally subtract the two resulting equations. Using (4.107) and (4.108), we get

$$J \sum_{j,k} \frac{1}{u_j(q_k - u_j)} + 2\pi(n+m) \sum_{j,k} \frac{1}{q_k - u_j} = 0, \tag{4.109}$$

or using (4.105)

$$J \sum_{j,k} \frac{1}{u_j(q_k - u_j)} = 4\pi^2 M_2 (mn + m^2). \tag{4.110}$$

Combining (4.104), (4.105), (4.106) and (4.110) then yields

$$\Delta_1(M_1, M_2, J; n, m) = \frac{\lambda}{2J^2}\Big[n^2 M_1\Big(1 - \frac{M_1}{J}\Big) + 2mnM_2\Big(1 - \frac{M_1}{J}\Big) + m^2 M_2\Big(1 - \frac{M_2}{J}\Big)\Big]. \tag{4.111}$$

If we compare (4.103) and (4.105), we get the constraint $pJ = nM_1 + mM_2$. Recalling that $M_2 = J_2$ and $M_1 = J_1 + J_2$ then gives

$$p(J_1 + J_2 + J_3) - n(J_1 + J_2) - mJ_2 = 0, \tag{4.112}$$

and this expression guides us to define new integers

$$m_1 \equiv p - n, \quad m_2 \equiv p - n - m, \quad m_3 \equiv p, \tag{4.113}$$

such that (4.112) becomes

$$m_1 J_1 + m_2 J_2 + m_3 J_3 = 0. \tag{4.114}$$

The expression in (4.111) can be rewritten in the same manner it was done in (4.37). Substituting n, m, M_1, and M_2 with the J_i's and m_i's defined in (4.113) and using (4.114), the anomalous dimension takes the form

$$\Delta_1(J_1, J_2, J_3; m_1, m_2, m_3) = \frac{\lambda}{2J}\sum_{i=1}^{3} m_i^2 \frac{J_i}{J}. \tag{4.115}$$

4.4 β-Deformed Spin Chains

We now turn to the β-deformed theory with the one-loop planar dilatation operator given by (3.91). Before we try to construct the Bethe equations for this operator, a relevant question is whether integrability is preserved in the deformed theory. To summarize, the hamiltonian of the β-deformed spin chain (with $\beta = \beta^* \equiv \gamma$) is given by

$$\mathcal{D}_\gamma^{Planar} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{J} H_i, \tag{4.116}$$

$$\begin{aligned} H_i = {} & E_{22}^i E_{33}^{i+1} + E_{33}^i E_{22}^{i+1} - e^{2\pi i\gamma} E_{23}^i E_{32}^{i+1} - e^{-2\pi i\gamma} E_{32}^i E_{23}^{i+1} \\ & + E_{33}^i E_{11}^{i+1} + E_{11}^i E_{33}^{i+1} - e^{2\pi i\gamma} E_{31}^i E_{13}^{i+1} - e^{-2\pi i\gamma} E_{13}^i E_{31}^{i+1} \\ & + E_{11}^i E_{22}^{i+1} + E_{22}^i E_{11}^{i+1} - e^{2\pi i\gamma} E_{12}^i E_{21}^{i+1} - e^{-2\pi i\gamma} E_{21}^i E_{12}^{i+1}. \end{aligned}$$

We note that the deformed hamiltonian (4.116) can be obtained from the undeformed by making the substitution $E_{ij}E_{kl} \to e^{i\pi(\epsilon_{ij}-\epsilon_{kl})\gamma} E_{ij}E_{kl}$ with ϵ antisymmetric and $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1$. This was exploited in [33], where it was shown that an R-matrix of the deformed spin chain can be constructed directly from the R-matrix of the undeformed spin chain using the simple relationship with the hamiltonian (4.3). Thus, if the undeformed spin chain admits the construction of an R-matrix satisfying the Yang-Baxter equation, so does the deformed spin chain, and integrability is preserved. Alternatively, one can make a position dependent change of base, which explicitly maps the deformed spin chain to an undeformed spin chain with twisted boundary conditions [34].

We will now derive the Bethe equations in the deformed subsector where only two complex fields are involved. We take the contributing fields to be X and Z and the dilatation operator (4.116) can then be written

$$\mathcal{D}_\gamma^{Planar} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{J} E_{XX}^i E_{ZZ}^{i+1} + E_{ZZ}^i E_{XX}^{i+1} - e^{i2\pi\gamma} E_{ZX}^i E_{XZ}^{i+1} - e^{-i2\pi\gamma} E_{XZ}^i E_{ZX}^{i+1}. \qquad (4.117)$$

If we can get rid of the phase factors in front of the last two terms, it is exactly the hamiltonian (4.2), and the results obtained above can be carried over. This is achieved through a position dependent change of basis. We denote the field at site k by $|\rangle_k$ and define the new states

$$|\tilde{X}\rangle_k \equiv e^{i2\pi k\gamma}|X\rangle_k, \qquad |\tilde{Z}\rangle_k \equiv |Z\rangle_k. \qquad (4.118)$$

The action of E_{XX} and E_{ZZ} on these new states is as on the old states, but the action E_{XZ} and E_{ZX} now gives rise to a phase:

$$E_{ZX}^k|\tilde{X}\rangle_k = e^{i2\pi k\gamma}E_{ZX}^k|X\rangle_k = e^{i2\pi k\gamma}|Z\rangle_k = e^{i2\pi k\gamma}|\tilde{Z}\rangle_k, \qquad (4.119)$$

$$E_{XZ}^k|\tilde{Z}\rangle_k = E_{XZ}^k|Z\rangle_k = |X\rangle_k = e^{-i2\pi k\gamma}|\tilde{X}\rangle_k. \qquad (4.120)$$

This leads us to define the new operators

$$\tilde{E}_{XX}^k = E_{XX}^k, \quad \tilde{E}_{ZZ}^k = E_{ZZ}^k, \quad \tilde{E}_{XZ}^k = e^{i2\pi k\gamma}E_{XZ}^k, \quad \tilde{E}_{ZX}^k = e^{-i2\pi k\gamma}E_{ZX}^k, \qquad (4.121)$$

with which the dilatation operator takes the form

$$\begin{aligned}\mathcal{D}_\gamma^{Planar} &= \frac{\lambda}{8\pi^2}\sum_{i=1}^{J} \tilde{E}_{XX}^i \tilde{E}_{ZZ}^{i+1} + \tilde{E}_{ZZ}^i \tilde{E}_{XX}^{i+1} - \tilde{E}_{ZX}^i \tilde{E}_{XZ}^{i+1} - \tilde{E}_{XZ}^i \tilde{E}_{ZX}^{i+1} \\ &= \frac{\lambda}{8\pi^2}\sum_{i=1}^{J}(1 - P_{i,i+1}). \end{aligned} \qquad (4.122)$$

This is exactly the ordinary spin chain hamiltonian (4.2) and it looks like we regained the $SU(2)$ symmetry that was originally broken in the super potential. This is only true locally on the spin chain, though. The boundary conditions get modified by the deformation and do not respect the $SU(2)$ symmetry, but this does not affect the integrability of the hamiltonian, so the spin chain remains integrable with the Bethe ansatz, S-matrix, and energy given by the same expressions as in the undeformed case.

Since we have identified $|X\rangle_k$ with $|X\rangle_{J+k}$ in the old basis, we have to make the identification

$$|\tilde{X}\rangle_k = e^{-i2\pi\gamma J}|\tilde{X}\rangle_{J+k}, \qquad (4.123)$$

in the new basis. This implies that we should impose the boundary conditions

$$\psi_{p_1 \dots p_2}(x_1, \dots, x_M) = e^{i2\pi J\gamma}\psi_{p_1 \dots p_2}(x_2, \dots, x_M, x_1 + J), \tag{4.124}$$

and the deformation thus introduces a phase factor in the Bethe equations

$$e^{-i2\pi\gamma J}\left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{4.125}$$

We also need to see how the cyclicity constraint changes with these twisted boundary conditions. The action of the translation operator is (in the new basis)

$$T|x_1, \dots, x_M\rangle = |x_1 + 1, \dots, x_M + 1\rangle e^{-i2\pi\gamma M}, \tag{4.126}$$

and we get the constraint

$$-2\pi\gamma M + \sum_{i=1}^{M} p_i = 0, \tag{4.127}$$

or if we exponentiate and express it in terms of Bethe roots

$$\prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} = e^{i2\pi\gamma M}. \tag{4.128}$$

When taking the thermodynamic limit ($J \to \infty$), we have to assume that $pJ \sim 1$ so the Bethe equations are not unstable to small variations in J. By the same argument, we will assume that $\gamma J \sim 1$.

Now, we proceed to find the rational solutions for the anomalous dimension of operators in the deformed gauge theory. Compared with the undeformed theory, the only difference is the phase factor appearing in (4.125) and (4.128). Taking the logarithm of these two equations and comparing with (4.28) and (4.29), we note that the resulting equations can be obtained from the corresponding undeformed equations by letting $n \to n - \gamma J$ and $m \to m + \gamma M$. Introducing the new integers m_1 and m_3 we see from equation (4.34)-(4.36) that this corresponds to the substitutions: $m_1 \to m_1 - \gamma J_3$ and $m_3 \to m_3 + \gamma J_1$ in the final expression (4.37), and the anomalous dimension in the deformed rational $SU(2)$ sector is thus

$$\Delta_1(J_1, J_3; m_1, m_3) = \frac{\lambda}{2J^2}\Big((m_1 - \gamma J_3)^2 J_1 + (m_3 + \gamma J_1)^2 J_3\Big). \tag{4.129}$$

We can apply the same procedure to the deformed $SU(3)$ spin chain, but the change of basis one has to make is a bit more involved [34]. Instead of going through these derivations, we will simply try to argue what the result for the anomalous dimension should be, using that the expression should reduce to (4.129) when $J_2 = 0$ and to (4.115) when $\gamma = 0$. Furthermore, the $SU(3)$ anomalous dimension should be invariant under cyclic permutation of the three indices and non-cyclic permutations accompanied by $\gamma \to -\gamma$ as can be seen by inspection of (3.90). The result should thus include the limits:

$$\Delta_1 = \frac{\lambda}{2J^2}\Big((m_1 - \gamma J_3)^2 J_1 + (m_3 + \gamma J_1)^2 J_3\Big), \qquad J_2 = 0, \tag{4.130}$$

$$\Delta_1 = \frac{\lambda}{2J^2}\Big((m_2 - \gamma J_1)^2 J_2 + (m_1 + \gamma J_2)^2 J_1\Big), \qquad J_3 = 0, \tag{4.131}$$

$$\Delta_1 = \frac{\lambda}{2J^2}\Big((m_3 - \gamma J_2)^2 J_3 + (m_2 + \gamma J_3)^2 J_2\Big), \qquad J_1 = 0. \tag{4.132}$$

Again, γ should enter as a phase in the Bethe equations, and the solution becomes linear in γ due to the logarithm. We thus expect that the anomalous dimension involves three terms of the form $(m_i + \gamma f_i(J_1, J_2, J_3))^2 J_i$. By the above symmetries, the functions f_i should fulfill

$$f_1(J_1, J_2, J_3) = f_2(J_2, J_3, J_1), \tag{4.133}$$

$$f_1(J_1, J_2, J_3) = -f_1(J_1, J_3, J_2), \tag{4.134}$$

so it will be enough to determine f_1. Clearly, the linear part of f_1 is $J_2 - J_3$ and the only quadratic term in accord with the above limits is $J_2 J_3$. However, such a term does not satisfy $f_1(J_1, J_2, J_3) = -f_1(J_1, J_3, J_2)$ and is thus excluded. We cannot exclude the presence of higher order terms though. For example, it would be consistent with the above constraints to include a term proportional to $J_3^2 J_2 - J_3 J_2^2$, but if we assume that f_1 does not contain cubic or higher order terms, we get the result

$$\Delta_\gamma(J_1, J_2, J_3; m_1, m_2, m_3) = \frac{\lambda}{2J}\sum_{i=1}^{3} \tilde{m}_i^2 \frac{J_i}{J}, \tag{4.135}$$

where

$$\tilde{m}_1 = m_1 + \gamma(J_2 - J_3), \quad \tilde{m}_2 = m_2 + \gamma(J_3 - J_1), \quad \tilde{m}_3 = m_3 + \gamma(J_1 - J_2). \tag{4.136}$$

In fact, this result *is* the correct generalization of (4.129) as was shown in [33]. The anomalous dimension was derived under the assumption of $\gamma J \sim 1$ and this ensures that (4.135) does not blow up in the thermodynamic limit. We will return to this at the end of section 5.4, where the result is reproduced from classical string theory.

5 String Theory

One of the main difficulties in testing the AdS/CFT correspondence is that it is still unknown how to quantize string theory on $AdS_5 \times S^5$. One can find classical solutions and calculate quantum corrections, but in general, these are only small in the limit where the 't Hooft coupling λ is large. To compare with the gauge theory results, it is desirable to find a sector where the classical results are valid even when λ is small.

Such solutions were found and analyzed by Frolov and Tseytlin in a sequence of papers [12, 13, 14]. They considered rigid circular strings spinning in $AdS_5 \times S^5$ and showed that quantum corrections to the classical energy are suppressed in the limit of large angular momenta. These solutions are exactly those that match the rational $SU(3)$ solutions from the last section and provide a nice one-loop verification of the AdS/CFT correspondence.

In this section we start by reviewing the basics of classical bosonic string theory. The equations of motion and certain constraints are derived from the Polyakov action and we introduce the notation used in the rest of this section. We then calculate the classical energy of Frolov-Tseytlin strings as a perturbation expansion in λ/J^2. The one-loop result exactly matches the anomalous dimension found in the gauge theory. We then follow the semiclassical quantization scheme of Frolov and Tseytlin and show that quantum corrections are suppressed in the limit of large angular momenta. Finally, we will sketch the arguments that let Lunin and Maldacena [15] to the gravity dual of β-deformed $\mathcal{N} = 4$ SYM and calculate the classical energy of strings in this deformed background. Again, the result of the previous section is reproduced.

5.1 Basics of Bosonic String Theory

The fundamental objects in string theory are one-dimensional strings and as such, the theory is very different from theories where zero-dimensional particles play the role as fundamental objects. Nevertheless, many of the well known concepts from particle physics can be a guideline to set up the theory of strings. One should distinguish theories of closed strings and open strings and in the following, we will only consider closed strings. More precisely, the theory that is conjectured to be dual to $\mathcal{N} = 4$ SYM is type IIB string theory, which is one of five distinct ten-dimensional superstring theories.

The action of a free relativistic particle can be taken as the proper time along the path traced out by the particle times a constant. The path of the particle is called a world-line and the proper time elapsed on the worldline multiplied by c, is the Lorentz invariant proper length of the world-line.

Since strings are one-dimensional objects, we expect them to trace out a two-dimensional world-sheet instead of a one-dimensional world-line, and a natural guess for the action of a free propagating string is just the area of this world-sheet times some constant. Whereas a world-line can be parameterized by a single parameter, we need two different parameters to parameterize a world-sheet and it is often convenient to choose a time-like parameter and a space-like parameter labeled by τ and σ, respectively. In d spacetime dimensions, the world-sheet can then be described by the d string coordinates $X^\mu(\tau, \sigma)$. This is the Nambu-Goto action and can be written [20]

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \tag{5.1}$$

where α' is the string length squared, and we have used the standard notation

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma}. \tag{5.2}$$

In addition to the obvious Lorentz invariance, the Nambu-Goto action is also independent of how we parameterize the string coordinates and we say it is reparameterization invariant.

The action can be simplified if one introduces a two-dimensional world-sheet metric $h^{\alpha\beta}$. With this metric, we can write a seemingly very different action. This is the Polyakov action:

$$S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{5.3}$$

where $h = det(h_{\alpha\beta})$ and the indices α and β run over τ and σ. We will take the world-sheet metric to have lorentzian signature $(-,+)$. If one derives the classical equations of motion for $h^{\alpha\beta}$ and inserts the result in the Polyakov action, we get the Nambu-Goto action back, and the actions are thus classically equivalent.

The Polyakov action is easier to work with than the Nambu-Goto action, since it has a simple quadratic dependence on the derivatives. The price to pay is of course the introduction of the new field $h^{\alpha\beta}$. In addition to the poincaré invariance and reparameterization invariance, the Polyakov action is also invariant under a two-dimensional Weyl transformation that acts on the variables as

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma), \tag{5.4}$$

$$h'_{\alpha\beta}(\tau,\sigma) = e^{2\omega(\tau,\sigma)} h_{\alpha\beta}(\tau,\sigma), \tag{5.5}$$

for arbitrary $\omega(\tau,\sigma)$. The reparameterization invariance can be used to locally set the off diagonal elements of the world-sheet metric to zero and the diagonal elements equal to each other. The metric then has the form $h_{\alpha\beta}(\tau,\sigma) = f^2(\tau,\sigma)\eta_{\alpha\beta}$ and is said to be conformally flat. The positive factor $f^2(\tau,\sigma)$ can be eliminated by a Weyl transformation. In the following, we will restrict ourselves to the class of parameterizations that result in a conformally flat world-sheet metric and we will use $\sqrt{-h}h_{\alpha\beta} = \eta_{\alpha\beta}$. This is called the conformal gauge.

If we use the conformal gauge and perform a variation of the Polyakov action with respect to the string coordinates, we get the wave equations

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0. \tag{5.6}$$

The variation with respect to the world-sheet metric gives

$$\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}(-\dot{X}^2 + X'^2) = 0. \tag{5.7}$$

This is really three equations, but only two are independent:

$$\dot{X} \cdot X' = 0, \tag{5.8}$$

$$\dot{X}^2 + X'^2 = 0, \tag{5.9}$$

and these are called the conformal gauge constraints. These constraints should always be imposed on solutions to the wave equations above and reflect the restriction to conformally flat world-sheet metrics.

It will be useful to think of the variables $X^\mu(\tau,\sigma)$ as independent fields instead of string coordinates parameterized by τ and σ. The Polyakov action then describes a two-dimensional field theory with d massless scalar fields X^μ. In this picture, the Poincaré invariance is an internal symmetry of the fields and the reparameterization invariance corresponds to general coordinate invariance. We should remember, though, that it is not a generalization of the usual field theories involving particles. The quantization of the Polyakov action corresponds to the ordinary quantum mechanics or a first quantization in the language of field theory.

To each of the symmetries in the action there is an associated conserved charge. The conserved charge associated with the general coordinate invariance is the energy-momentum tensor, and the conserved charges associated with Poincaré invariance are the momenta and Lorentz charges.

When the theory is quantized, a consistent theory requires a 26-dimensional spacetime for bosonic strings and ten-dimensional spacetime for superstrings. Although we only consider bosonic strings in the following, we will still think of them as being in a bosonic subsector of the full superstring theory and we thus only consider strings in ten dimensions.

The massless spectrum of closed bosonic strings can be divided in three parts corresponding to a symmetric Lorentz tensor $G_{\mu\nu}$, an antisymmetric Lorentz tensor $B_{\mu\nu}$, and a scalar field Φ. The states associated with $G_{\mu\nu}$ can be identified with gravitons, and string theory thus naturally contains gravity. $B_{\mu\nu}$ is called the Kalb-Ramond[15] field and is a natural generalization of the Maxwell potential A_μ when the "charged" objects carry to Lorentz indices. Φ is called the dilaton and is related to the string coupling g_s, but since we only consider non-interacting strings we will not discuss the dilaton further.

Since string theory gives rise to these fields, it is natural to consider strings moving in a gravity background (e.i. curved spacetime) and coupled to a background of the Kalb-Ramond field $B_{\mu\nu}$. The generalization of the Polyakov action when these backgrounds are included is [35]

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-h}\big[h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X^i) - \epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X^i)\big], \qquad (5.10)$$

where $\epsilon^{\alpha\beta}$ is antisymmetric with $\epsilon^{\tau\sigma} = 1$. Since the conformal gauge constraints were derived from the equations of motion for $h^{\alpha\beta}$, the Kalb-Ramond field will not affect these constraints, and the generalization of (5.8) and (5.9) in curved space is obtained by replacing the flat metric implicit in the scalar product in these expressions with $G_{\mu\nu}$. The equations of motion become more complicated, and the simple wave equation above is only true in flat space with cartesian coordinates.

[15]The term "Kalb-Ramond field" originates from pure bosonic string theory [20]. In superstring theory, the massless bosons are divided in two sectors: The Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector and the Ramond-Ramond (R-R) sector. In type IIB string theory, both sectors give rise to a two-form potential, but the field we have in mind, when referring to $B_{\mu\nu}$, is the NS-NS field.

5.2 Strings in $AdS_5 \times S^5$

We will consider strings in $AdS_5 \times S^5$ with common radius of curvature R. Both AdS_5 and S^5 are maximally symmetric spaces and thus have 15 isometries each corresponding to the generators of the isometry group $SO(2,4) \times SO(6)$. We can embed the five-dimensional anti-de Sitter space in a six-dimensional pseudo-euclidian space with coordinates Y_M, $M \in \{0,1,2,3,4,5\}$ and the five-sphere in six-dimensional euclidian space with coordinates X_A, $A \in \{1,2,3,4,5,6\}$. These coordinates should be of order R, but it will be more practical to work with normalized string coordinates instead, where a factor of R has been extracted. The string coordinates are then subject to the (normalized) constraints

$$Y_M Y_M \equiv Y_0^2 - Y_1^2 - Y_2 - Y_3 - Y_4 + Y_5^2 = 1, \tag{5.11}$$

$$X_A X_A \equiv X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 1. \tag{5.12}$$

There is a standard parameterization of AdS_5 and S^5 that solves these constraints:

$$Y_0 = \cosh\rho \cos t, \qquad X_1 = \sin\alpha \cos\theta \cos\phi_1, \tag{5.13}$$

$$Y_1 = \sinh\rho \cos\psi \sin\vartheta_1, \qquad X_2 = \sin\alpha \cos\theta \sin\phi_1, \tag{5.14}$$

$$Y_2 = \sinh\rho \cos\psi \cos\vartheta_1, \qquad X_3 = \sin\alpha \sin\theta \cos\phi_2, \tag{5.15}$$

$$Y_3 = \sinh\rho \sin\psi \sin\vartheta_2, \qquad X_4 = \sin\alpha \sin\theta \sin\phi_2, \tag{5.16}$$

$$Y_4 = \sinh\rho \sin\psi \cos\vartheta_2, \qquad X_5 = \cos\alpha \cos\phi_3, \tag{5.17}$$

$$Y_5 = \cosh\rho \sin t, \qquad X_6 = \cos\alpha \sin\phi_3. \tag{5.18}$$

The ten angles (including the dimensionless time) appearing here are functions of τ and σ and can be taken as the fundamental fields characterizing the string propagating on $AdS_5 \times S^5$. One could now go on and calculate the metric of $AdS_5 \times S^5$ from these equations and then write the Polyakov action in terms of the angular string coordinates. However, we prefer to work with the embedding fields a bit longer, since it will make the connection to the conserved charges of $\mathcal{N} = 4$ SYM more clear.

5.2.1 Embedding Coordinates

We will label the conserved charges of AdS_5 by S_{MN}, $M, N \in \{0,1,2,3,4,5\}$ and the conserved charges of S^5 by J_{AB}, $A, B \in \{1,2,3,4,5,6\}$. These are then given by [20]

$$S_{MN} = \sqrt{\lambda}\mathcal{S}_{MN} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} (Y_M \dot{Y}_N - Y_N \dot{Y}_M) d\sigma, \tag{5.19}$$

$$J_{AB} = \sqrt{\lambda}\mathcal{J}_{AB} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} (X_A \dot{X}_B - X_B \dot{X}_A) d\sigma, \tag{5.20}$$

where $\sqrt{\lambda} = R^2/\alpha'$.

We will now look at rigid strings located at the center of AdS_5 and rotating in S^5 with three different angular momenta. We make the following ansatz for the target space fields:

$$t = \kappa\tau, \qquad \rho = 0, \qquad \psi = \vartheta_1 = \vartheta_2 = 0, \qquad \alpha = \alpha_0, \quad \theta = \theta_0, \tag{5.21}$$

$$\phi_1 = \omega_1\tau + m_1\sigma, \qquad \phi_2 = \omega_2\tau + m_2\sigma, \qquad \phi_3 = \omega_3\tau + m_3\sigma, \tag{5.22}$$

where α_0 and θ_0 are independent of τ and σ. Since we are dealing with closed strings we should require periodic boundary conditions on the world-sheet. This means that $X_A(\tau, \sigma) = X_A(\tau, \sigma + 2\pi)$, and we thus require the m_i's to be integers. The m_i's counts the number of times the string is wound around S^5 in the angular direction of ϕ_i and are called winding numbers. If all the m_i's are zero, the string will be point-like. The ω_i's are the frequencies of the spinning string and are proportional to the angular momenta of the string.

The non-vanishing conserved charges are then

$$\mathcal{E} = \mathcal{S}_{05} = \kappa, \tag{5.23}$$

$$\mathcal{J}_1 = \mathcal{J}_{12} = \sin^2\alpha_0 \cos^2\theta_0 \omega_1, \tag{5.24}$$

$$\mathcal{J}_2 = \mathcal{J}_{34} = \sin^2\alpha_0 \sin^2\theta_0 \omega_2, \tag{5.25}$$

$$\mathcal{J}_3 = \mathcal{J}_{56} = \cos^2\alpha_0 \omega_3. \tag{5.26}$$

The energy has been identified with S_{05}, since it turns out to be the generator of time translations when one writes the the AdS part of the action in global coordinates. Our task is now to express the energy in terms of the three angular momenta.

Before we go on, we note that if we define the three complex target space fields

$$X \equiv X_1 + iX_2 = \sin\alpha \cos\theta e^{i\phi_1}, \tag{5.27}$$

$$Y \equiv X_3 + iX_4 = \sin\alpha \sin\theta e^{i\phi_2}, \tag{5.28}$$

$$Z \equiv X_5 + iX_6 = \cos\alpha e^{i\phi_3}, \tag{5.29}$$

the string charges $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ are precisely those that correspond to the generators of translations in the phases of X, Y, and Z, respectively. These generators span the Cartan subalgebra of $SO(6)$ and a quantized string can be labeled by the quantum numbers associated with these generators as well as the energy of the string. Of course, it is not a coincident we have chosen the letters X, Y, and Z just as we did with the complex scalar fields in $\mathcal{N} = 4$ SYM. Strings of the type (5.21)-(5.22) are exactly expected to correspond to operators consisting of complex scalar fields in the gauge theory.

To proceed, we need the equations of motion for the string and conformal gauge constraints. The S^5 constraint (5.12) can be included in the Polyakov action by use of a Lagrange multiplier field Λ. Using the conformal gauge, we get

$$S_{S^5} = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Big[\partial_a X_A \partial^a X_A + \Lambda(X^2 - 1)\Big], \quad \sqrt{\lambda} = \frac{R^2}{\alpha'}. \tag{5.30}$$

The equation of motion for Λ will then give back the constraint (5.12), and the equations of motion for the string coordinates are

$$-\partial^\alpha \partial_\alpha X_A + \Lambda X_A = 0. \tag{5.31}$$

These equations are satisfied if we take

$$\Lambda = \omega_1^2 - m_1^2 = \omega_2^2 - m_2^2 = \omega_3^2 - m_3^2. \tag{5.32}$$

We see that with this kind of ansatz, all the m_i and ω_i need to be independent if we want a solution with three independent angular momenta. We could easily have picked a simpler ansatz than the one applied above. One could for example take two of the winding numbers to be equal, but then the associated frequencies would also have to be (numerically) equal and the associated angular momenta would be proportional.

When imposing the conformal gauge constraints (5.8) and (5.9), we should include Y_0 and Y_5 as well as all the S^5 embedding string coordinates and remember that Y_0 comes with a minus due to the Lorentz signature of Minkowski space. Inserting our ansatz, gives the equation

$$\mathcal{E}^2 = \kappa^2 = (\omega_1^2 + m_1^2)\sin^2\alpha_0\cos^2\theta_0 + (\omega_2^2 + m_2^2)\sin^2\alpha_0\sin^2\theta_0 + (\omega_3^2 + m_3^2)\cos^2\alpha_0, \quad (5.33)$$

where we used (5.23) in the first equality. The second conformal gauge constraint (5.5) gives

$$\omega_1 m_1 \sin^2\alpha_0\cos^2\theta_0 + \omega_2 m_2 \sin^2\alpha_0\sin^2\theta_0 + \omega_3 m_3\cos^2\alpha_0 = 0. \quad (5.34)$$

This equation can be combined with (5.24)-(5.26) to give the constraint

$$m_1 J_1 + m_2 J_2 + m_3 J_3 = 0, \quad (5.35)$$

which is equivalent to (5.34). The energy is now a function of $(\omega_1, \omega_2, \omega_3, m_1, m_2, m_3, \alpha_0, \theta_0)$, and we would like to express it in terms of angular momenta and winding numbers: $E = E(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3; m_1, m_2, m_3)$. The equations (5.32) can be regarded as 2 constraints relating $\mathcal{J}_i$, m_i, $\sin^2\alpha_0$ and $\sin^2\theta_0$, and these can be used to eliminate $\sin^2\alpha_0$ and $\sin^2\theta_0$ from the energy once the ω_i's are expressed in terms of $\mathcal{J}_i$, α_0, and θ_0 and inserted in (5.33).

It is not possible to obtain a closed expression for the energy, but if we define $\mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$. it can be written as a perturbation series in $\mathcal{J}^{-2}$. We shall assume that all ω_i and thus all J_i are non-negative[16] and that $\omega_i^2 > m_i^2$. We then define $\nu^2 \equiv \omega_i^2 - m_i^2$ and the energy can be written

$$\begin{aligned}\mathcal{E}^2 &= 2\omega_1^2\sin^2\alpha_0\cos^2\theta_0 + 2\omega_2^2\sin^2\alpha_0\sin^2\theta_0 + 2\omega_3^2\cos^2\alpha_0 - \nu^2 \\ &= 2\sum_{i=1}^{3}\omega_i\mathcal{J}_i - \nu^2 = 2\sum_{i=1}^{3}\sqrt{m_i^2+\nu^2}\,\mathcal{J}_i - \nu^2. \end{aligned} \quad (5.36)$$

Equations (5.24)-(5.26) and (5.32) can then be combined to yield

$$\sum_{i=1}^{3}\frac{\mathcal{J}_i}{\omega_i} = \sum_{i=1}^{3}\frac{\mathcal{J}_i}{\sqrt{m_i^2+\nu^2}} = 1. \quad (5.37)$$

The strategy is now to solve this equation, giving ν as a function of m_i and $\mathcal{J}_i$, and then insert this into (5.36), which should then be supplemented by the constraint (5.35).

[16] This is a very natural assumption for comparison with the gauge theory, since there, the R-charges simply count the number of complex scalars in the operators.

If $\nu^2 > m_i^2$, equation (5.37) can be written as a Taylor series in the quantity m_i^2/ν^2

$$|\nu| = \sum_{i=1}^{3} \frac{\mathcal{J}_i}{\sqrt{1+\frac{m_i^2}{\nu^2}}} = \sum_{i=1}^{3} \mathcal{J}_i \Big(1 - \frac{m_i^2}{2\nu^2} + \frac{3m_i^4}{8\nu^4} - \dots\Big)$$

$$= \mathcal{J}\Big(1 - \frac{1}{2\nu^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{3}{8\nu^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} - \dots\Big). \tag{5.38}$$

We can then derive the following expressions

$$\nu^2 = \mathcal{J}^2\Big[1 - \frac{1}{\nu^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{3}{4\nu^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{1}{4\nu^4}\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 + \dots\Big], \tag{5.39}$$

$$\frac{1}{|\nu|} = \frac{1}{\mathcal{J}}\Big[1 + \frac{1}{2\nu^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} - \frac{3}{8\nu^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{1}{4\nu^4}\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 + \dots\Big], \tag{5.40}$$

$$\frac{1}{\nu^2} = \frac{1}{\mathcal{J}^2}\Big[1 + \frac{1}{\nu^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} - \frac{3}{4\nu^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{3}{4\nu^4}\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 + \dots\Big]. \tag{5.41}$$

The last expression makes it clear that (5.38), (5.39) and (5.40) become power series in $\mathcal{J}^{-2}$. We will now write these expressions in terms of $\mathcal{J}$, explicitly including terms up to second order in $\mathcal{J}^{-2}$:

$$|\nu| = \mathcal{J}\Big[1 - \frac{1}{2\mathcal{J}^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} - \frac{1}{2\mathcal{J}^4}\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 + \frac{3}{8\mathcal{J}^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} + \dots\Big], \tag{5.42}$$

$$\nu^2 = \mathcal{J}^2\Big[1 - \frac{1}{\mathcal{J}^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} - \frac{3}{4\mathcal{J}^4}\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 + \frac{3}{4\mathcal{J}^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} + \dots\Big], \tag{5.43}$$

$$\frac{1}{|\nu|} = \frac{1}{\mathcal{J}}\Big[1 + \frac{1}{2\mathcal{J}^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{3}{4\mathcal{J}^4}\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 + \frac{3}{8\mathcal{J}^4}\sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}} + \dots\Big]. \tag{5.44}$$

The energy can also be written as an expansion in ν^{-2}

$$\mathcal{E}^2 = 2\sum_{i=1}^{3}\sqrt{m_i^2+\nu^2}\,\mathcal{J}_i - \nu^2 = 2|\nu|\sum_{i=1}^{3}\mathcal{J}_i\Big(1 + \frac{m_i^2}{2\nu^2} - \frac{m_i^4}{8\nu^4} + \dots\Big) - \nu^2,$$

and using (5.42), (5.43) and (5.44), we get

$$\mathcal{E}^2 = \mathcal{J}^2\left(1 + \frac{1}{\mathcal{J}^2}\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \frac{1}{4\mathcal{J}^4}\Big[\Big(\sum_{i=1}^{3} m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}}\Big)^2 - \sum_{i=1}^{3} m_i^4 \frac{\mathcal{J}_i}{\mathcal{J}}\Big] + \dots\right), \tag{5.45}$$

and thus

$$E \equiv \lambda\mathcal{E} = J\Big(1 + \frac{\lambda}{2J^2}\sum_{i=1}^{3} m_i^2 \frac{J_i}{J} - \frac{\lambda^2}{8J^4}\sum_{i=1}^{3} m_i^4 \frac{J_i}{J} + \dots\Big), \qquad \sum_{i=1}^{3} m_i J_i = 0, \tag{5.46}$$

where we have expanded the square root and reintroduced $J_i = \sqrt{\lambda}\mathcal{J}_i$. The energy takes the form of a perturbation expansion in an effective coupling constant λ/J^2 and the second term is exactly what was found for the anomalous dimension in the gauge theory (4.115). The chiral primary operator $Tr[Z^J]$ corresponds to a point-like string with energy $E = J$ which propagates on the equator parameterized by ϕ_3.

We should pause to consider the range of validity of this result. A priori, one might expect to match the classical results and quantum corrections loop by loop on both sides of the AdS/CFT correspondence. But equation (5.46) is a purely classical result, and the one-loop *quantum* correction to the conformal dimension in the gauge theory is thus contained in the *classical* energy of spinning strings. On the other hand, we cannot really expect to match the classical energy of spinning strings with *perturbative* results from the gauge theory, since it is the full quantized string theory that is conjectured to be dual to $\mathcal{N} = 4$ SYM. Quantum corrections to the string energy are only assumed to be small in the limit where $\lambda \gg 1$ as we will show in the next subsection. However, the gauge theory results were derived in the perturbative regime where we assumed that $\lambda \ll 1$, so how come the results match after all? The reason is that for this particular kind of solutions, quantum corrections are suppressed in the limit of $J \gg 1$ [12, 13, 14]. This will be discussed in section 5.3.

Another point is that strictly speaking, the R-charges only take integer values in the spin chain result (4.115), whereas the angular momenta in (5.46) can take any positive values. In the quantized string theory, the J_i's are quantized, but in the large J semi-classical limit where we are comparing calculations, the integer nature of the J_i's is not important.

5.2.2 The Metric of $AdS_5 \times S^5$

In the next section, we will consider a deformed gravity background and we cannot use the simple embedding coordinates used above. Therefore, we will briefly show how the above results can be obtained using the metric of $AdS_5 \times S^5$ and the ten angular fields appearing in (5.13)-(5.18).

The metric of $AdS_5 \times S^5$ can be induced from the parametrization (5.13)-(5.18) and is given by

$$ds^2 = R^2\big(ds^2_{AdS_5} + ds^2_{S^5}\big) = R^2\big(G^{AdS_5}_{mn}dy^m dy^n + G^{S^5}_{ab}dx^a dx^b\big), \tag{5.47}$$

$$ds^2_{AdS_5} = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\psi^2 + \cos^2\psi d\vartheta_1^2 + \sin^2\psi d\vartheta_2^2), \tag{5.48}$$

$$ds^2_{S^5} = d\alpha^2 + \cos^2\alpha d\phi_3^2 + \sin^2\alpha(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2), \tag{5.49}$$

where we have denoted the five AdS_5 angles by y^n and the five S^5 angles by x^a. The Polyakov action becomes

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma L, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'} \tag{5.50}$$

$$L = \sqrt{-h}h^{\alpha\beta}\big(G^{AdS_5}_{mn}\partial_\alpha y^m\partial_\beta y^n + G^{S^5}_{ab}\partial_\alpha y^a\partial_\beta y^b\big). \tag{5.51}$$

In the following, we will again use the conformal gauge where $\sqrt{-h}h^{\alpha\beta} = \text{diag}(-1,1)$. We start by noting that the AdS_5 part is invariant to translations in ϑ_1, ϑ_2, and t, and the S^5 part is invariant to translations in the three fields ϕ_1, ϕ_2, and ϕ_3. These isometries have associated conserved charges, which are those that correspond to the Cartan generators of $SO(2,4) \times SO(6)$. In general, if the translation $\varphi_i \to \varphi_i + \varepsilon$ is a symmetry of the action, there is a conserved charge given by

$$Q_i = -\frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma \frac{\partial L}{\partial \dot{\varphi}_i}, \tag{5.52}$$

where φ_i labels any of the fields appearing in the action. In the present case, we get six conserved charges $(E, S_1, S_2, J_1, J_2, J_3)$, which are given by

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \cosh^2\rho\, \dot{t} \equiv \sqrt{\lambda}\mathcal{E}, \tag{5.53}$$

$$S_1 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \sinh^2\rho \sin^2\psi \dot{\vartheta}_1 \equiv \sqrt{\lambda}\mathcal{S}_1, \tag{5.54}$$

$$S_2 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \sinh^2\rho \cos^2\psi \dot{\vartheta}_2 \equiv \sqrt{\lambda}\mathcal{S}_2, \tag{5.55}$$

$$J_1 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \sin^2\alpha \cos^2\theta \dot{\phi}_1 \equiv \sqrt{\lambda}\mathcal{J}_1, \tag{5.56}$$

$$J_2 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \sin^2\alpha \sin^2\theta \dot{\phi}_2 \equiv \sqrt{\lambda}\mathcal{J}_2, \tag{5.57}$$

$$J_3 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \cos^2\alpha \dot{\phi}_3 \equiv \sqrt{\lambda}\mathcal{J}_3. \tag{5.58}$$

The AdS_5 charges S_1 and S_2 were defined as S_{12} and S_{34} in embedding coordinates.

We will again consider strings located at the center of AdS_5 moving in S^5 and use the ansatz (5.21)-(5.22). The non-vanishing conserved charges are given by (5.23)-(5.26), and the only non-trivial equations of motion are those of α and θ:

$$\alpha: \quad \omega_3^2 - m_3^2 + \cos^2\theta_0(-\omega_1^2 + m_1^2) + \sin^2\theta_0(-\omega_2^2 + m_2^2) = 0, \tag{5.59}$$

$$\theta: \quad \omega_1^2 - m_1^2 - \omega_2^2 + m_2^2 = 0, \tag{5.60}$$

which are seen to be satisfied if we take

$$\omega_1^2 - m_1^2 = \omega_2^2 - m_2^2 = \omega_3^2 - m_3^2. \tag{5.61}$$

The conformal gauge constraints now read

$$G_{mn}^{AdS_5}\dot{y}^m y'^n + G_{ab}^{S^5}\dot{x}^a x'^b = 0, \tag{5.62}$$

$$G_{mn}^{AdS_5}(\dot{y}^m\dot{y}^n + y'^m y'^n) + G_{ab}^{S^5}(\dot{x}^a\dot{x}^b + x'^a x'^b) = 0, \tag{5.63}$$

and yields the two equations

$$\sin^2\alpha_0\cos^2\theta_0\omega_1 m_1 + \sin^2\alpha_0\sin^2\theta_0\omega_2 m_2 + \cos^2\alpha_0\omega_3 m_3 = 0, \tag{5.64}$$
$$\sin^2\alpha_0\cos^2\theta_0(\omega_1^2 + m_1^2) + \sin^2\alpha_0\sin^2\theta_0(\omega_2^2 + m_2^2) + \cos^2\alpha_0(\omega_3^2 + m_3^2) = \kappa^2. \tag{5.65}$$

The equations (5.61), (5.64), and (5.65) are precisely those obtained with embedding coordinates (5.32), (5.34), and (5.33) and the expression for the energy can now be derived as in (5.36)-(5.46).

5.3 Quantum Corrections

As mentioned above, we can only trust the classical calculations in string theory when $\lambda \gg 1$. This is well-known from quantum field theory where the partition function is proportional to $\exp(\frac{i}{\hbar}S)$ and the classical limit is obtained by taking the limit $\hbar \to 0$. When $S \gg \hbar$, the partition function will fluctuate violently and the only contributing field configurations are those which give rise to a stationary value of the action: the classical field configurations.

In terms of Feynman diagrams, the classical approximation is an expansion in tree diagrams. The loop diagrams become quantum corrections and are proportional to $\hbar^n$, where n counts the number of loops. When considering the string action, the situation is similar, but the loop counting parameter is now $1/\sqrt{\lambda}$, and one should thus be able to write quantum corrections to the energy as a power expansion in $1/\sqrt{\lambda}$. We already saw that the classical expression for the energy could be written $E = \sqrt{\lambda}\mathcal{E}_0$, where $\mathcal{E}_0$ only depended on the "couplingless" charges $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ in addition to the three winding numbers. To recapitulate, we had that

$$\mathcal{E}_0 = \mathcal{J}\big(1 + \frac{a_1^{(0)}}{\mathcal{J}^2} + \frac{a_2^{(0)}}{\mathcal{J}^4} + \ldots\big), \tag{5.66}$$

where $a_1^{(0)}$ and $a_2^{(0)}$ were given in (5.46). Since $\sqrt{\lambda}$ only appears as a factor in front of the action it could have been anticipated from the start that the classical expression for $\mathcal{E}$ cannot involve λ. The same is expected to be true for the quantum corrections, and the energy should thus take the form

$$E_{tot} = \sqrt{\lambda}\mathcal{E}_0 + \mathcal{E}_1 + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2 + \frac{1}{\lambda}\mathcal{E}_3 + \ldots = \sum_{n=0}^{\infty} E_n, \tag{5.67}$$

where $\mathcal{E}_n$ is independent of λ. Superficially, this seems devastating for a comparison with perturbative results in the gauge theory. The anomalous dimensions in the last section were calculated under the assumption that $\lambda \ll 1$, but string quantum corrections seem to blow up in this limit.

There is, however, one more limit that needs to be taken into account. We were only able to solve the Bethe equations when we took the thermodynamic limit of long spin chains. In string theory, this translates into fast spinning strings with $J \gg 1$ and such strings are expected to behave semiclassically. The hope is now that somehow the classical behavior due to large angular momentum will dominate the quantum behavior due to small λ.

This is exactly what happens if we assume that $\mathcal{E}_n$ takes the form [36]

$$\mathcal{E}_n = \frac{a_1^{(n)}}{\mathcal{J}^{n+1}} + \frac{a_2^{(n)}}{\mathcal{J}^{n+3}} + \frac{a_3^{(n)}}{\mathcal{J}^{n+5}} \dots, \tag{5.68}$$

which is inspired by the classical piece $\mathcal{E}_0$ (neglecting the leading $\mathcal{J}$). The n-loop piece of (5.67) will then be given by

$$E_n = \frac{1}{(\sqrt{\lambda})^{n-1}}\mathcal{E}_n = \frac{a_1^{(n)}\lambda}{J^{n+1}} + \frac{a_2^{(n)}\lambda^2}{J^{n+3}} + \frac{a_3^{(n)}\lambda^3}{J^{n+5}} \dots, \tag{5.69}$$

where $J = \sqrt{\lambda}\mathcal{J}$ has been reintroduced. The full expression (5.67) can now be reorganized into the double expansion

$$E_{tot} = J\Big[1 + \sum_{k=1}^{\infty}\left(\frac{\lambda}{J^2}\right)^k \sum_{n=0}^{\infty}\frac{a_k^{(n)}}{J^n}\Big]. \tag{5.70}$$

If we take the limit of $J \gg 1$ with λ/J^2 fixed, we see that quantum corrections are suppressed and the energy becomes a perturbation expansion in λ/J^2. One may then conjecture that the result can be interpolated to weak coupling allowing a direct comparison with perturbative gauge theory results [14].

The general circumstances under which (5.68) is true remain to be clarified, but below we will sketch how to obtain the one-loop correction to the classical energy for our particular solution and show that it is indeed suppressed by J^{-1} compared to the classical energy. We follow the procedure of [12, 13, 14] and consider a small perturbation of the rational classical solution found above. The zero point energy of this perturbation are then the first order quantum correction to the energy and can be found by summing all characteristic frequencies of the fluctuation fields. However, first we need to analyze the stability of the classical solution under perturbations.

5.3.1 Stability

It is important that the classical solution is stable under small perturbations, since otherwise, quantum fluctuations could result in a transition to a different state and the classical result would not be reliable. The lagrangian for a string spinning on S^5 is given by

$$L = \partial_a X_A \partial^a X_A + \Lambda(X_A X_A - 1), \tag{5.71}$$

and we will now consider small fluctuations $\tilde{X_A}$ near the classical solution

$$X = X_1 + iX_2 = \sin\alpha_0 \cos\theta_0 e^{i(\omega_1\tau + m_1\sigma)}, \tag{5.72}$$

$$Y = X_3 + iX_4 = \sin\alpha_0 \sin\theta_0 e^{i(\omega_2\tau + m_2\sigma)}, \tag{5.73}$$

$$Z = X_5 + iX_6 = \cos\alpha_0 e^{i(\omega_3\tau + m_3\sigma)}, \tag{5.74}$$

$$\Lambda = \nu^2 = \omega_i^2 - m_i^2. \tag{5.75}$$

We replace the fields in L by $X_A \to X_A + \tilde{X}_A$ and $\Lambda \to \Lambda + \tilde{\Lambda}$ and insert the solutions for X_A and Λ above. We then obtain the "constrained" lagrangian

$$\tilde{L}_{con} = \partial_a X_A \partial^a X_A + \partial_a \tilde{X}_A \partial^a \tilde{X}_A + 2\tilde{\Lambda} X_A \tilde{X}_A + \Lambda \tilde{X}_A \tilde{X}_A + \tilde{\Lambda}_A \tilde{X}_A \tilde{X}_A, \tag{5.76}$$

where we used partial integration to eliminate a mixed derivative term. The first term is just a constant and is ignored in the following. We will also disregard the third order term, since this just gives rise to small corrections to the free spectrum of the fluctuation fields[17]. The equation of motion for $\tilde{\Lambda}$ is $X_A \tilde{X}_A = 0$, and it will be easiest to first solve this constraint and then substitute into the "unconstrained" quadratic lagrangian

$$\tilde{L} = \partial_a \tilde{X}_A \partial^a \tilde{X}_A + \Lambda \tilde{X}_A \tilde{X}_A. \tag{5.77}$$

The fields $\tilde{X}_A$ are seen to be massive with mass $\sqrt{\omega_i^2 - m_i^2}$. To simplify the constraint, we write the perturbation fields as

$$\tilde{X} = \tilde{X}_1 + i\tilde{X}_2 = e^{i(\omega_1 \tau + m_1 \sigma)}(g_1 + if_1), \tag{5.78}$$

$$\tilde{Y} = \tilde{X}_3 + i\tilde{X}_4 = e^{i(\omega_2 \tau + m_2 \sigma)}(g_2 + if_2), \tag{5.79}$$

$$\tilde{Z} = \tilde{X}_5 + i\tilde{X}_6 = e^{i(\omega_3 \tau + m_3 \sigma)}(g_3 + if_3), \tag{5.80}$$

and the constraint $X_A \tilde{X}_A = 0$ reduces to

$$\sum_{i=1}^{3} a_i g_i = 0, \qquad \sum_{i=1}^{3} a_i = 1, \tag{5.81}$$

where we have defined

$$a_1 \equiv \sin\alpha_0 \cos\theta_0, \quad a_2 \equiv \sin\alpha_0 \sin\theta_0, \quad a_3 \equiv \cos\alpha_0. \tag{5.82}$$

The unconstrained lagrangian can now be expressed in terms of the real fields f_i and g_i giving

$$L = \sum_{i=1}^{3} \Big[-\dot{f}_i^2 - \dot{g}_i^2 + f_i'^2 + g_i'^2 + 2\omega_i f_i \dot{g}_i - 2\omega_i g_i \dot{f}_i - 2m_i f_i g_i' + 2m_i g_i f_i' \Big], \tag{5.83}$$

which is to be supplemented with the constraint (5.81). We proceed by isolating g_3 from (5.81) and inserting it into the lagrangian, which then depends on five independent fields.

As a side remark, we note that the resulting lagrangian represents a two-dimensional massive scalar field theory coupled to a two-dimensional non-abelian gauge field. To see

[17] The fluctuation fields are small by assumption, and we could have introduced a small bookkeeping parameter to make this explicit. Corrections from the third order term would then become a perturbation expansion in this parameter.

this, it is convenient to define the five-dimensional vector $z_p = (f_1, f_2, f_3, g_1, g_2)$, with which the lagrangian takes the form

$$L = -M_{pq}\dot{z}_p\dot{z}_q + M_{pq}z'_p z'_q - 2F_{pq}z_p\dot{z}_q + 2G_{pq}z_p z'_q, \tag{5.84}$$

where F_{pq} and G_{pq} are antisymmetric matrices and M_{pq} is symmetric. We lost the diagonal structure in the kinetic terms because we eliminated g_3 from the lagrangian, but since M_{pq} is symmetric, it can be diagonalized by an orthogonal transformation, which does not change the antisymmetry of F_{pq} and G_{pq}. Performing the diagonalization and rescaling the fields so that M becomes the identity, the lagrangian can be written (keeping the same notation for transformed fields and matrices)

$$L = -(\dot{z}_p - F_{pq}z_q)^2 + (z'_p - G_{pq}z_q)^2 - (F_{pq}F_{qk} - G_{pq}G_{qk})z_p z_k. \tag{5.85}$$

The two components of the gauge field are thus $(A_0, A_1) = (F_{pq}, G_{pq})$ and are functions of (a_i, m_i, ω_i), but independent of τ and σ. The mass matrix $F_{pq}F_{qk} - G_{pq}G_{qk}$ is symmetric. There is no fields strength term so we could regard F_{pq} and G_{pq} as external fields supplying a mass for the scalar fields.

To analyze the stability of the fluctuations, we first derive the equations of motion for the five scalar fields (using (5.83)):

$$\begin{aligned}
&-\ddot{f}_1 + f''_1 = 2\omega_1\dot{g}_1 - 2m_1 g'_1,\\
&-\ddot{f}_2 + f''_2 = 2\omega_2\dot{g}_2 - 2m_2 g'_2,\\
&-\ddot{f}_3 + f''_3 = -2\omega_3(a_1\dot{g}_1 + a_2\dot{g}_2) + 2m_3(a_1 g'_1 + a_2 g'_2),\\
&-(1+a_1^2)\ddot{g}_1 - a_1 a_2\ddot{g}_2 + 2\omega_1\dot{f}_1 - 2\omega_3 a_1\dot{f}_3 + (1+a_1^2)g''_1 + a_1a_2 g''_2 - 2m_1 f'_1 + 2m_3 a_1 f'_3 = 0,\\
&-(1+a_2^2)\ddot{g}_2 - a_1 a_2\ddot{g}_1 + 2\omega_2\dot{f}_2 - 2\omega_3 a_2\dot{f}_3 + (1+a_2^2)g''_2 + a_1a_2 g''_1 - 2m_2 f'_2 + 2m_3 a_2 f'_3 = 0,
\end{aligned} \tag{5.86}$$

where we have set $a_3 = 1$ for the moment (it can always be reintroduced by the substitutions $a_1 \to a_1/a_3$, $a_2 \to a_2/a_3$). To solve these equations, we note that the solution should be periodic in σ, and the σ dependence is extracted as a standard Fourier decomposition. We then collect the five scalar fields as the vector $\mathbf{z}$ (defined above) and make the following ansatz for the solution:

$$\mathbf{z}(\tau, \sigma) = \sum_{n=-\infty}^{\infty} \mathbf{A}_n e^{i(\Omega_n\tau + n\sigma)}. \tag{5.87}$$

Such a solution usually works when one is dealing with coupled linear differential equations. The derivatives just bring down factors of Ω_n and n and the result is five algebraic equations that determine the vector $\mathbf{A}_n$ for a given n. These equations can be written as the matrix equation $M_{pq}A_q = 0$, where $M_{pq} = M_{pq}(m_i, \omega_i, a_i, n, \Omega_n)$. We are interested in the nontrivial solutions where $\mathbf{A}_n \neq 0$, which is equivalent to requiring $detM = 0$. This condition will be used to determine the characteristic frequencies Ω_n. Unstable solutions are characterized by Ω_n having an imaginary part, so we will require that all Ω_n are real for a solution to be stable. The determinant of the matrix M will be some polynomial in Ω_n and we therefore get different sets of solutions $\{\Omega_n^{(k)}\}$. When dealing with linear differential equations, a sum of particular solutions will always be a solution and the general solution for $\mathbf{z}$ should thus be a superposition of the particular solutions (5.87), each being characterized by a set of characteristic frequencies $\{\Omega_n^{(k)}\}$. In the following, we will suppress the indices on Ω. Inserting (5.87) into the equations of motion, results in the matrix:

$$M = \begin{pmatrix} (\Omega^2 - n^2) & 0 & 0 & -i\tilde{\Omega}_1 & 0 \\ 0 & (\Omega^2 - n^2) & 0 & 0 & -i\tilde{\Omega}_2 \\ 0 & 0 & (\Omega^2 - n^2) & ia_1\tilde{\Omega}_3 & ia_2\tilde{\Omega}_3 \\ i\tilde{\Omega}_1 & 0 & -ia_1\tilde{\Omega}_3 & (1+a_1^2)(\Omega^2 - n^2) & a_1a_2(\Omega^2 - n^2) \\ 0 & i\tilde{\Omega}_2 & -ia_2\tilde{\Omega}_3 & a_1a_2(\Omega^2 - n^2) & (1+a_2^2)(\Omega^2 - n^2) \end{pmatrix},$$

where $\tilde{\Omega}_i = 2(\omega_i\Omega - m_i n)$. Reintroducing the a_3 dependence, the equation $det(M) = 0$ can now be written

$$(\Omega^2 - n^2)\Big[(\Omega^2 - n^2)^4 - (\Omega^2 - n^2)^2[(a_2^2 + a_3^3)\tilde{\Omega}_1^2 + (a_1^2 + a_3^3)\tilde{\Omega}_2^2 + (a_1^2 + a_2^3)\tilde{\Omega}_3^2] \\ + a_1^2\tilde{\Omega}_2^2\tilde{\Omega}_3^2 + a_2^2\tilde{\Omega}_3^2\tilde{\Omega}_1^2 + a_3^2\tilde{\Omega}_1^2\tilde{\Omega}_2^2\Big] = 0. \quad (5.88)$$

We are not interested in actually finding the solutions for $\mathbf{A}_n$ corresponding to a certain Ω, but we would like to determine the range of parameters for which Ω_n is real implying a stable solution. In addition to the solution with $\Omega^2 = n^2$, there is eight solutions for Ω. It is not possible to give the general range of parameters for which all these roots are real, but we can consider certain limits to see how it works. For example, the solution with two equal non-vanishing spins $J_1 = J_3$, $J_2 = 0$, which is the result of choosing $a_1^2 = a_3^2 = \frac{1}{2}$, $m_1 = -m_3 = m$, $\omega_1 = \omega_3 = \omega$, $m_2 = \omega_2 = a_2 = 0$, gives rise to the equation

$$(\Omega^2 - n^2)^2 - 4(\omega^2\Omega^2 + n^2m^2) = 0, \quad (5.89)$$

and we are led to the stability condition that $n^2 < 4m^2$. Thus, since we are summing over all n, the general solution corresponding to this choice of parameters is unstable.

However, it is possible to analyze the stability for more general values of the parameters in the large $\mathcal{J}$ limit and one can find a range of parameters for which the solutions are stable. This was done in [14] for a certain three spin solution with $J_1 = J_2$.

5.3.2 One-loop Correction

The hamiltonian of a quadratic potential is that of a harmonic oscillator. From ordinary quantum mechanics, we know that the bosonic harmonic oscillator hamiltonian is given by

$$\hat{H} = \omega(\hat{N} + \frac{1}{2}), \quad (5.90)$$

where $\hat{N}$ gives the excitation number of a state. If we are dealing with fermions, the plus should be replaced by a minus. The ground state energy is obtained by setting $\hat{N} = 0$. In quantum field theory there is an oscillator for each mode of the fields and the hamiltonian is thus a sum over terms like (5.90).

The fluctuations around the classical string solution can be regarded as such harmonic oscillators and the ground state energy is the first order quantum correction to the classical solution. We can find this correction by summing over all characteristic frequencies of the fluctuations, which can be found in by a procedure similar to the way we found the Ω_n's above. However, not all of the frequencies will contribute and one has to carefully analyze the lagrangian of the fluctuation fields including contributions from AdS_5. Such an analysis was carried out in [38], where it was shown how to obtain the characteristic frequencies in a family of field theories including (5.85). In d spacetime dimensions, there is $d-2$ independent sets of characteristic frequencies. In ten-dimensional string theory, there is thus eight sets of characteristic frequencies which contribute to the energy. Classically, we are allowed to consider only bosonic strings, but when considering quantum fluctuations, we should allow for fermionic fluctuations as well, since we know that this is really a subsector of a supersymmetric theory. We should thus include both fermionic and bosonic characteristic frequencies in the calculation, and the groundstate of the world-sheet energy of the fluctuation fields can be found by the following expression [14]

$$E_{2d}^{(0)} = \frac{1}{2}\Big(\sum_{n\in\mathbb{Z}} \omega_n^B - \sum_{r\in\mathbb{Z}+\frac{1}{2}} \omega_r^F\Big), \tag{5.91}$$

where $\omega_n^B = \sum_{k=1}^{8} \Omega_{n,k}^B$ and $\omega_r^F = \sum_{k=1}^{8} \Omega_{r,k}^F$. The world-sheet energy is the generator of translations in the world-sheet time τ, whereas the spacetime energy is the generator of translations in Minkowski time t. Due to our parametrization $t = \kappa t$, there is a simple relationship between these, and the first correction to the spacetime energy is

$$E_1 = E_{2d}^{(0)} = \frac{1}{2\kappa}\Big(\sum_{n\in\mathbb{Z}} \omega_n^B - \sum_{r\in\mathbb{Z}+\frac{1}{2}} \omega_r^F\Big). \tag{5.92}$$

The calculation can be performed in certain limits but is quite involved, so we simply state the leading term found in [14] for a three-spin solution with $J_1 = J_2$:

$$E_1 = \frac{e_1}{\kappa^2}, \tag{5.93}$$

where e_1 is of order 1. Expanding this in the limit of large $\mathcal{J}$ with κ given by (5.45), gives the result

$$E_1 = a_1 \frac{1}{\mathcal{J}^2} + a_2 \frac{1}{\mathcal{J}^4} + \ldots = a_1 \frac{\lambda}{J^2} + a_2 \frac{\lambda^2}{J^4} + \ldots. \tag{5.94}$$

This result is exactly the assumption we needed for the one-loop quantum correction to be suppressed in the large J limit (5.68).

As we have just argued that, quantum corrections are suppressed in the limit of large

J. On the other hand, much could be learned if we could figure out how to match these corrections with results from the gauge theory. Clearly, the large J semiclassical limit of string theory should correspond to the long spin chain limit of the gauge theory and the string quantum corrections become finite size corrections in spin chain calculations.

We can obtain a concrete expression for a contribution to the first order correction in string theory by considering the particularly simple state in the $SU(2)$ sector ($J_2 = 0$) with $J_1 = J_3$ [37]. In that case, the bosonic characteristic frequencies is given by (5.89) and we get

$$\Omega_{\pm}^2 = n^2 + 2\nu^2 + 2m^2 \pm 2\sqrt{(\nu^2 + m^2)^2 + n^2(\nu^2 + 2m^2)}, \tag{5.95}$$

where we used that $\omega^2 = \nu^2 + m^2$. We then expand Ω_- in large ν and obtain

$$\Omega_-^{(\pm)} = \pm\frac{n}{2\nu}\sqrt{n^2 - 4m^2} + \mathcal{O}(\frac{1}{\nu^3}). \tag{5.96}$$

For this configuration, we have $\kappa^2 = \nu^2 + 2m^2$, and the contribution from these oscillations to the one loop energy correction is then

$$E_n = \frac{1}{\kappa}|\Omega_-| = \frac{n}{2\nu^2}\sqrt{n^2 - 4m^2} + \mathcal{O}(\frac{1}{\nu^4}) = \frac{\lambda}{2J^2}n\sqrt{n^2 - 4m^2} + \mathcal{O}(\frac{\lambda^2}{J^4}), \tag{5.97}$$

with $J = J_1 + J_2 = 2J_1$. We note again that the expression fulfils the scaling property (5.68) that allows us to discard the contribution in the limit of large J.

This result with $m = 1$ was reproduced from the gauge theory in [39]. The authors considered a state with an equal number of X's and Z's such that $J_1 = J_2 = J/2$. We recall that there is a number of possible solutions for the anomalous dimension for such a state, corresponding to different distributions of the Bethe roots in the complex plane. One of these distributions gives rise to the rational expression for the anomalous dimension found in the previous section, and this particular distribution should in some sense be dual to the string ansatz (5.21)-(5.22). The authors then calculated the change in anomalous dimension corresponding to a small change in the particular root distribution, and the result exactly matches the quantum correction (5.97). Therefore, it seems that the Bethe roots and thus the magnons associated with spin chains play a fundamental role in the matching of string states with gauge theory operators. The Bethe roots condense on "stringy" curves in the complex plane and the fluctuations of these curves correspond to quantum fluctuations of actual strings in $AdS_5 \times S^5$.

5.4 Lunin-Maldacena Background

Since it has been conjectured that the gravity dual of $\mathcal{N} = 4$ SYM is exactly $AdS_5 \times S^5$, it is natural to ask whether one can find the gravity dual of the β-deformed theory (3.87). To summarize, the original theory had an $SU(4)$ R-symmetry that was broken in the deformed theory. The deformed theory is still conformal invariant and it is therefore expected that the gravity dual of the deformation should only affect the S^5 part of the metric. The deformed theory had the superpotential

$$W_\gamma = Tr[e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2], \tag{5.98}$$

with a $U(1) \times U(1)$ non-R-symmetry that acts by

$$1: \qquad (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, e^{i\alpha_1}\Phi_2, e^{-i\alpha_1}\Phi_3), \tag{5.99}$$

$$2: \qquad (\Phi_1, \Phi_2, \Phi_3) \to (e^{-i\alpha_2}\Phi_1, e^{i\alpha_2}\Phi_2, \Phi_3). \tag{5.100}$$

The gravity dual of this theory was given in a paper by Lunin and Maldacena [15]. The two $U(1)$ symmetries play a prominent role and in fact, their arguments hold true for any field theory with a $U(1)\times U(1)$ global symmetry. Obviously, the undeformed theory, which is obtained by setting $\gamma = 0$, also has the same $U(1) \times U(1)$ symmetry, and the symmetry thus survives the deformation.

The prescription of Lunin and Maldacena is to identify the two-torus of the undeformed metric that corresponds to the $U(1)\times U(1)$ symmetry in the field theory and then perform a certain $SL(2, R)$ transformation on that two-torus. The transformation is

$$\tau \equiv B_{12} + i\sqrt{g} \to \tau_\gamma = \frac{\tau}{1+\gamma\tau}, \tag{5.101}$$

where B_{12} is the Kalb-Ramond field along the directions of the two $U(1)$ symmetries and $\sqrt{g}$ is the volume of the two-torus.[18]

We will now briefly review their arguments, although a proper treatment requires an understanding of the relationship between string theory and noncommutative geometry in field theories. In fact, their main argument is largely based on a paper by Seiberg and Witten [40] and goes as follows. Consider an open string theory with some two-torus in the metric and a D-brane sitting at a point where the two-torus shrinks to zero size. In the low energy limit, there will be a gauge theory living on the brane and the two $U(1)$ symmetries will act as a global symmetry in this theory. According to [40], the effect of the transformation (5.101) will be a change of the product between fields living on the brane:

$$\Phi_i\Phi_j \to e^{i\pi\gamma(Q_i^1Q_j^2 - Q_i^2Q_j^1)}\Phi_i\Phi_j. \tag{5.102}$$

Here Q_i^1 and Q_i^2 are the charges of the field Φ_i under the two $U(1)$ symmetries mentioned above. In the superpotential (5.98), each term contains three fields and the trace implies that there is a cyclic multiplication between them. This means that in both terms, all three fields are multiplying each other, but in opposite order. The deformed superpotential should thus be obtained from the undeformed superpotential by

$$Tr\Phi_1\Phi_2\Phi_3 \to e^{i\pi\gamma(Q_1^1Q_2^2 - Q_1^2Q_2^1 + Q_2^1Q_3^2 - Q_2^2Q_3^1 + Q_3^1Q_1^2 - Q_3^2Q_1^1)}Tr\Phi_1\Phi_2\Phi_3, \tag{5.103}$$

$$Tr\Phi_1\Phi_3\Phi_2 \to e^{i\pi\gamma(Q_1^1Q_3^2 - Q_1^2Q_3^1 + Q_3^1Q_2^2 - Q_3^2Q_2^1 + Q_2^1Q_1^2 - Q_2^2Q_1^1)}Tr\Phi_1\Phi_2\Phi_3. \tag{5.104}$$

[18]One should use a dimensionless volume in this formula. In our case, we use the effective dimensionless radius of curvature as the factor in front of the metric e.i. $R^2 \to R^2/\alpha' = \sqrt{\lambda}$.

We now observe that if we denote the charge of the superfield Φ_i by $\bar{Q} = (Q^1, Q^2)_i$ and assign the charges $(0, -1/3)_1$, $(1/3, 1/3)_2$, and $(-1/3, 0)_3$ to the superfields in accordance with (5.99) and (5.100), we exactly obtain (5.98) from the transformation rules (5.103) and (5.104).

The three superfields each have an associated complex scalar field and there is three $U(1)$ transformations of these fields that correspond to isometries of the three angular fields ϕ_i in the metric of S^5 (5.49). However, the two-torus we are looking for should correspond to the two $U(1)$ symmetries above, which apparently do not generate simple translations in any of the ϕ_i's. For example, the first $U(1)$ translates ϕ_2 and ϕ_3 by the same amount, but in opposite directions. If we instead define the fields φ_i by

$$\phi_1 = \varphi_3 - \varphi_2, \quad \phi_2 = \varphi_1 + \varphi_2 + \varphi_3, \quad \phi_3 = \varphi_3 - \varphi_1, \tag{5.105}$$

$$\varphi_1 = \frac{1}{3}(\phi_1 + \phi_2 - 2\phi_3), \quad \varphi_2 = \frac{1}{3}(\phi_2 + \phi_3 - 2\phi_1), \quad \varphi_3 = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3), \tag{5.106}$$

then $U(1)_1$ generates translations in φ_1, $U(1)_2$ generates translations in φ_2, and φ_3 is left invariant by both transformations. If we were to define three complex fields W_i with phases corresponding to the fields φ_i, we would assign to them the $U(1) \times U(1)$ charges $(1/3, 0)_1$, $(0, 1/3)_2$, and $(0, 0)_3$.

The metric (5.49) can be expressed in terms of the φ_i's, and the volume of the two-torus parameterized by φ_1 and φ_2 can then be found [15]:

$$\sqrt{g} = \sqrt{\lambda \sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha \sin^2 \theta \cos^2 \theta)}. \tag{5.107}$$

Since we have previously considered an $AdS_5 \times S^5$ background with no Kalb-Ramond field, the transformation (5.101) simplifies a bit:

$$\tau = i\sqrt{g} \rightarrow \tau_\gamma = \frac{i\sqrt{g}}{1 + i\gamma\sqrt{g}} = (B_{12})_\gamma + i\sqrt{g_\gamma}, \tag{5.108}$$

$$(B_{12})_\gamma = \frac{\gamma g}{1 + \gamma^2 g}, \qquad \sqrt{g_\gamma} = \frac{\sqrt{g}}{1 + \gamma^2 g}. \tag{5.109}$$

One can now find the metric of the transformed two-torus and express the result in terms of the old angular fields ϕ_1, ϕ_2, and ϕ_3. The result is [15]

$$\begin{aligned} ds^2_{S^5_\gamma} = \quad & d\alpha^2 + \sin^2 \alpha d\theta^2 + G\left[\cos^2 \alpha d\phi_3^2 + \sin^2 \alpha \cos^2 \theta d\phi_1^2 + \sin^2 \alpha \sin^2 \theta d\phi_2^2\right] \\ & + \tilde{\gamma}^2 G \sin^4 \alpha \cos^2 \alpha \sin^2 \theta \cos^2 \theta \big(d\phi_1 + d\phi_2 + d\phi_3\big)^2, \end{aligned} \tag{5.110}$$

where $\tilde{\gamma} = \sqrt{\lambda}\gamma$ and

$$G = \frac{1}{1 + \tilde{\gamma}^2 \sin^2 \alpha \left[\cos^2 \alpha + \sin^2 \alpha \sin^2 \theta \cos^2 \theta\right]}. \tag{5.111}$$

In terms of the old angular fields, the Kalb-Ramond field no longer has just one component. It is most easily expressed as the differential form

$$B = \sqrt{\lambda}\tilde{\gamma} G \sin^2\alpha \Big(\sin^2\alpha \sin^2\theta \cos^2\theta d\phi_1 d\phi_2 + \cos^2\alpha \sin^2\theta d\phi_2 d\phi_3 + \cos^2\alpha \cos^2\theta d\phi_3 d\phi_1 \Big),$$

and using (5.10), we get the S^5 part of the action for the gravity dual of β-deformed $\mathcal{N} = 4$:

$$\begin{aligned} S_\gamma = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \Big\{ & \sqrt{-h} h^{\alpha\beta} \Big[-\partial_\alpha t \partial_\beta t + \partial_\alpha \alpha \partial_\beta \alpha + \sin^2\alpha \partial_\alpha \theta \partial_\beta \theta \\ & + G \cos^2\alpha \partial_\alpha \phi_3 \partial_\beta \phi_3 + G \sin^2\alpha \cos^2\theta \partial_\alpha \phi_1 \partial_\beta \phi_1 + G \sin^2\alpha \sin^2\theta \partial_\alpha \phi_2 \partial_\beta \phi_2 \\ & + \tilde{\gamma}^2 G \sin^4\alpha \cos^2\alpha \sin^2\theta \cos^2\theta \Big(\partial_\alpha \phi_1 + \partial_\alpha \phi_2 + \partial_\alpha \phi_3 \Big) \Big(\partial_\beta \phi_1 + \partial_\beta \phi_2 + \partial_\beta \phi_3 \Big) \Big] \\ & - 2\tilde{\gamma} G \epsilon^{\alpha\beta} \sin^2\alpha \Big(\sin^2\alpha \sin^2\theta \cos^2\theta \partial_\alpha \phi_1 \partial_\beta \phi_2 + \cos^2\alpha \sin^2\theta \partial_\alpha \phi_2 \partial_\beta \phi_3 \\ & + \cos^2\alpha \cos^2\theta \partial_\alpha \phi_3 \partial_\beta \phi_1 \Big) \Big\}. \end{aligned} \tag{5.112}$$

As mentioned earlier, the AdS_5-part of the action does not change under the deformation and is still given by (5.48). Translations in ϕ_1, ϕ_2, and ϕ_3 are still isometries of the metric, and using (5.52), we can calculate the conserved charges associated with these:

$$\begin{aligned} J_1 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma G \Big[& \sin^2\alpha \cos^2\theta \dot{\phi}_1 + \tilde{\gamma}^2 \sin^4\alpha \cos^2\alpha \sin^2\theta \cos^2\theta \big(\dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3 \big) \\ & + \gamma \big(\sin^4\alpha \sin^2\theta \cos^2\theta \phi_2' - \sin^2\alpha \cos^2\alpha \cos^2\theta \phi_3' \big) \Big], \end{aligned} \tag{5.113}$$

$$\begin{aligned} J_2 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma G \Big[& \sin^2\alpha \sin^2\theta \dot{\phi}_2 + \tilde{\gamma}^2 \sin^4\alpha \cos^2\alpha \sin^2\theta \cos^2\theta \big(\dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3 \big) \\ & + \gamma \big(-\sin^4\alpha \sin^2\theta \cos^2\theta \phi_1' + \sin^2\alpha \cos^2\alpha \sin^2\theta \phi_3' \big) \Big], \end{aligned} \tag{5.114}$$

$$\begin{aligned} J_3 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma G \Big[& \cos^2\alpha \dot{\phi}_3 + \tilde{\gamma}^2 \sin^4\alpha \cos^2\alpha \sin^2\theta \cos^2\theta \big(\dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3 \big) \\ & + \gamma \big(-\sin^2\alpha \cos^2\alpha \sin^2\theta \phi_2' + \sin^2\alpha \cos^2\alpha \cos^2\theta \phi_1' \big) \Big]. \end{aligned} \tag{5.115}$$

If we once again restrict ourselves to strings located at the center of AdS_5, the conformal gauge constraints become

$$\begin{aligned} & -\dot{t}t' + \dot{\alpha}\alpha' + \sin^2\alpha \dot{\theta}\theta' + G \Big[\sin^2\alpha \cos^2\theta \dot{\phi}_1 \phi_1' + \sin^2\alpha \sin^2\theta \dot{\phi}_2 \phi_2' + \cos^2\alpha \dot{\phi}_3 \phi_3' \Big] \\ & + \tilde{\gamma}^2 G \sin^4\alpha \cos^2\alpha \sin^2\theta \cos^2\theta \big(\dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3 \big) \big(\phi_1' + \phi_2' + \phi_3' \big) = 0, \end{aligned} \tag{5.116}$$

and

$$\begin{aligned} & -\dot{t}^2 - t'^2 + \dot{\alpha}^2 + \alpha'^2 + \sin^2\alpha (\dot{\theta}^2 + \theta'^2) \\ & + G \Big[\sin^2\alpha \cos^2\theta (\dot{\phi}_1^2 + \phi_1'^2) + \sin^2\alpha \sin^2\theta (\dot{\phi}_2^2 + \phi_2'^2) + \cos^2\alpha (\dot{\phi}_3^2 + \phi_3'^2) \Big] \\ & + \tilde{\gamma}^2 G \sin^4\alpha \cos^2\alpha \sin^2\theta \cos^2\theta \Big[\big(\dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3 \big)^2 + \big(\phi_1' + \phi_2' + \phi_3' \big)^2 \Big] = 0. \end{aligned} \tag{5.117}$$

These equations along with the equations of motion determine the classical energy of strings as a function of the J_i's.

5.4.1 Spinning Strings in Deformed S^5

We will again consider solutions with $t = \kappa\tau$, $\phi_i = \omega_i\tau + m_i\sigma$, $\alpha = \alpha_0$ and $\theta = \theta_0$. The equations of motion for the ϕ_i's will then be trivially satisfied. With $x \equiv \sin\alpha_0$ and $y \equiv \sin\theta_0$, the equations of motion for θ and α are

$$\begin{aligned}
&-(-\omega_1^2+m_1^2)+(-\omega_2^2+m_2^2)\\
&+\tilde{\gamma}^2x^2(1-x^2)(1-2y^2)\Big[-\big(\omega_1+\omega_2+\omega_3\big)^2+\big(m_1+m_2+m_3\big)^2\Big]\\
&-2\tilde{\gamma}\Big[x^2(1-2y^2)(\omega_1m_2-\omega_2m_1)+(1-x^2)(\omega_2m_3-\omega_3m_2+\omega_1m_3-\omega_3m_1)\Big]\\
&-\tilde{\gamma}^2Gx^2(1-2y^2)F(x,y,m_i,\omega_i)=0,
\end{aligned}\tag{5.118}$$

and

$$\begin{aligned}
&\omega_3^2-m_3^2+(1-y^2)(-\omega_1^2+m_1^2)+y^2(-\omega_2^2+m_2^2)\\
&+\tilde{\gamma}^2x^2(2-3x^2)y^2(1-y^2)\Big[-\big(\omega_1+\omega_2+\omega_3\big)^2+\big(m_1+m_2+m_3\big)^2\Big]\\
&-2\tilde{\gamma}\Big[2x^2y^2(1-y^2)(\omega_1m_2-\omega_2m_1)\\
&\qquad+(1-2x^2)y^2(\omega_2m_3-\omega_3m_2)+(1-2x^2)(1-y^2)(\omega_3m_1-\omega_1m_3)\Big]\\
&-\tilde{\gamma}^2G\Big[(1-2x^2)+2x^2y^2(1-y^2)\Big]F(x,y,m_i,\omega_i)=0,
\end{aligned}\tag{5.119}$$

respectively, where

$$\begin{aligned}
F(x,y,m_i,\omega_i)=&(1-x^2)(-\omega_3^2+m_3^2)+x^2(1-y^2)(-\omega_1^2+m_1^2)+x^2y^2(-\omega_2^2+m_2^2)\\
&+\tilde{\gamma}^2x^4(1-x^2)y^2(1-y^2)\Big[-\big(\omega_1+\omega_2+\omega_3\big)^2+\big(m_1+m_2+m_3\big)^2\Big]\\
&-2\tilde{\gamma}\Big[x^4y^2(1-y^2)(\omega_1m_2-\omega_2m_1)+x^2(1-x^2)y^2(\omega_2m_3-\omega_3m_2)\\
&+x^2(1-x^2)(1-y^2)(\omega_3m_1-\omega_1m_3)\Big].
\end{aligned}\tag{5.120}$$

The conserved charge associated with time translations does not change when the metric is deformed and we thus still have $\mathcal{E} = \kappa$. We can then use (5.117) to express the energy in terms of the rest of the fields:

$$\begin{aligned}
\mathcal{E}^2=\kappa^2=\;&G\Big[x^2(1-y^2)(\omega_1^2+m_1^2)+x^2y^2(\omega_2^2+m_2^2)+(1-x^2)(\omega_3^2+m_3^2)\Big]\\
&+\tilde{\gamma}^2Gx^4(1-x^2)y^2(1-y^2)\Big[\big(\omega_1+\omega_2+\omega_3\big)^2+\big(m_1+m_2+m_3\big)^2\Big].
\end{aligned}\tag{5.121}$$

The constraint (5.116) can be written

$$m_1J_1+m_2J_2+m_3J_3=0,\tag{5.122}$$

which is equal to the constraint with $\gamma = 0$.

The situation is now similar to the undeformed case. The energy is given by the conformal gauge constraint and the equations of motion can be used to eliminate α_0 and θ_0 once we have expressed ω_i in terms of J_i and inserted these in (5.121). However, in the undeformed theory, the equations of motions could be reduced to the simple equations $\omega_i^2 - m_i^2 = \nu^2$ and we could then express the energy in terms of ν, which was determined perturbatively. The situation at hand is much more complicated, since the equations of motion are given by (5.118) and (5.119), which are not as easy to handle.

Fortunately, it is not as bad as it looks. The energy and equations of motion can be rewritten in a form that explicitly shows how the expression for the energy (5.46) changes under the deformation.

First we need to express the frequencies ω_i in terms of the angular momenta. With the ansatz stated above we get

$$\mathcal{J}_1 = G\Big[\sin^2\alpha_0\cos^2\theta_0\omega_1 + \tilde{\gamma}^2\sin^4\alpha_0\cos^2\alpha_0\sin^2\theta_0\cos^2\theta_0\big(\omega_1+\omega_2+\omega_3\big) + \tilde{\gamma}\big(\sin^4\alpha_0\sin^2\theta_0\cos^2\theta_0 m_2 - \sin^2\alpha_0\cos^2\alpha_0\cos^2\theta_0 m_3\big)\Big], \tag{5.123}$$

$$\mathcal{J}_2 = G\Big[\sin^2\alpha_0\sin^2\theta_0\omega_2 + \tilde{\gamma}^2\sin^4\alpha_0\cos^2\alpha_0\sin^2\theta_0\cos^2\theta_0\big(\omega_1+\omega_2+\omega_3\big) + \tilde{\gamma}\big(-\sin^4\alpha_0\sin^2\theta_0\cos^2\theta_0 m_1 + \sin^2\alpha_0\cos^2\alpha_0\sin^2\theta_0 m_3\big)\Big], \tag{5.124}$$

$$\mathcal{J}_3 = G\Big[\cos^2\alpha_0\omega_3 + \tilde{\gamma}^2\sin^4\alpha_0\cos^2\alpha_0\sin^2\theta_0\cos^2\theta_0\big(\omega_1+\omega_2+\omega_3\big) + \tilde{\gamma}\big(-\sin^2\alpha_0\cos^2\alpha_0\sin^2\theta_0 m_2 + \sin^2\alpha_0\cos^2\alpha_0\cos^2\theta_0 m_1\big)\Big], \tag{5.125}$$

and these equations can be inverted to yield

$$\omega_1 = \frac{\mathcal{J}_1}{x^2(1-y^2)} + \tilde{\gamma}^2\Big[(1-x^2)(\mathcal{J}_1-\mathcal{J}_2) + x^2y^2(\mathcal{J}_1-\mathcal{J}_3)\Big] + \tilde{\gamma}\Big[(1-x^2)m_3 - x^2y^2m_2\Big],$$
$$\omega_2 = \frac{\mathcal{J}_2}{x^2y^2} + \tilde{\gamma}^2\Big[(1-x^2)(\mathcal{J}_2-\mathcal{J}_1) + x^2(1-y^2)(\mathcal{J}_2-\mathcal{J}_3)\Big] + \tilde{\gamma}\Big[x^2(1-y^2)m_1 - (1-x^2)m_3\Big],$$
$$\omega_3 = \frac{\mathcal{J}_3}{(1-x^2)} + \tilde{\gamma}^2\Big[x^2y^2(\mathcal{J}_3-\mathcal{J}_1) + x^2(1-y^2)(\mathcal{J}_3-\mathcal{J}_2)\Big] + \tilde{\gamma}\Big[x^2y^2m_2 - x^2(1-y^2)m_1\Big].$$

In addition, we note that the sum of the frequencies are simply given by

$$\omega_1+\omega_2+\omega_3 = \frac{\mathcal{J}_1}{x^2(1-y^2)} + \frac{\mathcal{J}_2}{x^2y^2} + \frac{\mathcal{J}_3}{1-x^2}. \tag{5.126}$$

We now have to insert the frequencies into (5.118) and (5.119). If we start with the function

$F(x,y,m_i,\omega_i)$ appearing in both equations, it can be written

$$F(x,y,m_i,\omega_i) = G^{-1}\Big\{x^2(1-y^2)m_1^2 + x^2y^2m_2^2 + (1-x^2)m_3^2 - \frac{\mathcal{J}_1^2}{x^2(1-y^2)} - \frac{\mathcal{J}_2^2}{x^2y^2} - \frac{\mathcal{J}_3^2}{1-x^2}$$
$$-\tilde{\gamma}^2\Big[(1-x^2+x^2y^2)\mathcal{J}_1^2 + (1-x^2y^2)\mathcal{J}_2^2 + x^2\mathcal{J}_3^2 - 2(1-x^2)\mathcal{J}_1\mathcal{J}_2$$
$$- 2x^2y^2\mathcal{J}_1\mathcal{J}_3 - 2x^2(1-y^2)\mathcal{J}_2\mathcal{J}_3\Big]\Big\},$$

and the G's thus cancel from the equations of motion. When the ω_i's are inserted in the full expressions, it is possible to factor out another G^{-1} from both equations of motion. The equation for θ reduces to the expression

$$\Big(\frac{\mathcal{J}_1}{x^2(1-y^2)}\Big)^2 - \Big(\frac{\mathcal{J}_2}{x^2y^2}\Big)^2 + \tilde{\gamma}^2\Big[\mathcal{J}_1^2 - \mathcal{J}_2^2 + 2\mathcal{J}_2\mathcal{J}_3 - 2\mathcal{J}_1\mathcal{J}_3\Big]$$
$$+ m_2^2 - m_1^2 + 2\tilde{\gamma}\Big[(m_1+m_2)\mathcal{J}_3 - m_1\mathcal{J}_2 - m_2\mathcal{J}_1\Big] = 0,$$

and the equation for α becomes

$$\frac{\mathcal{J}_1^2}{x^4(1-y^2)} + \frac{\mathcal{J}_2^2}{x^4y^2} - \frac{\mathcal{J}_3^2}{(1-x^2)^2} - (1-y^2)m_1^2 - y^2m_2^2 + m_3^2$$
$$- 2\tilde{\gamma}\Big[(1-y^2)m_1(\mathcal{J}_2-\mathcal{J}_3) + y^2m_2(\mathcal{J}_3-\mathcal{J}_1) - m_3(\mathcal{J}_1-\mathcal{J}_2)\Big]$$
$$+ \tilde{\gamma}^2\Big[(1-y^2)\mathcal{J}_1^2 + y^2\mathcal{J}_2^2 - \mathcal{J}_3^2 + 2(1-y^2)\mathcal{J}_2\mathcal{J}_3 + 2y^2\mathcal{J}_1\mathcal{J}_3 - 2\mathcal{J}_1\mathcal{J}_2\Big] = 0.$$

We insert the ω_i's in (5.121) and get the energy

$$\begin{aligned}\mathcal{E}^2 =& \frac{\mathcal{J}_1^2}{x^2(1-y^2)} + \frac{\mathcal{J}_2^2}{x^2y^2} + \frac{\mathcal{J}_3^2}{1-x^2} + x^2(1-y^2)m_1^2 + x^2y^2m_2^2 + (1-x^2)m_3^2\\ &+ 2\tilde{\gamma}\Big[x^2(1-y^2)(\mathcal{J}_2-\mathcal{J}_3)m_1 + x^2y^2(\mathcal{J}_3-\mathcal{J}_1)m_2 + (1-x^2)(\mathcal{J}_1-\mathcal{J}_2)m_3\Big]\\ &+ \tilde{\gamma}^2\Big[(1-x^2+x^2y^2)\mathcal{J}_1^2 + (1-x^2y^2)\mathcal{J}_2^2 + x^2\mathcal{J}_3^2 - 2(1-x^2)\mathcal{J}_1\mathcal{J}_2\\ &- 2x^2y^2\mathcal{J}_1\mathcal{J}_3 - 2x^2(1-y^2)\mathcal{J}_2\mathcal{J}_3\Big]. \end{aligned} \tag{5.127}$$

Now we make a very relieving observation. The energy and the two equations of motion can be obtained from the corresponding equations in the undeformed background (5.45), (5.59), (5.60), by substituting

$$m_1 \to m_1 + \tilde{\gamma}(\mathcal{J}_2 - \mathcal{J}_3), \quad m_2 \to m_2 + \tilde{\gamma}(\mathcal{J}_3 - \mathcal{J}_1), \quad m_3 \to m_3 + \tilde{\gamma}(\mathcal{J}_1 - \mathcal{J}_2), \tag{5.128}$$

and thus, the energy can be calculated by the exact same procedure as we did in the undeformed background. Defining $\tilde{m}_1 = m_1 + \gamma(J_2 - J_3)$ and so on, the energy simply becomes

$$E = J\Big(1 + \frac{\lambda}{2J^2}\sum_{i=1}^{3}\tilde{m}_i^2\frac{J_i}{J} - \frac{\lambda^2}{8J^4}\sum_{i=1}^{3}\tilde{m}_i^4\frac{J_i}{J} + \dots\Big), \qquad \sum_{i=1}^{3} m_iJ_i = 0. \tag{5.129}$$

The second term agrees with the one-loop gauge theory result obtained from the deformed $SU(3)$ spin chain (4.135). As we have seen, the deformed spin chain could be mapped to an undeformed spin chain with twisted boundary conditions. The result here is similar. The energy of strings in the deformed background can be obtained from strings in the undeformed background by giving the winding numbers a momentum dependent "twist". We note that the point-like string dual to the chiral primary $Tr[Z^J]$, is not affected by the deformation.

To compare with the gauge theory, we need to consider the strong curvature limit with $\lambda \ll 1$ and $J \gg 1$. For generic values of γ, the energy (5.129) becomes a perturbation series in λ rather than λ/J^2, since the $\tilde{m}_i$'s scale as J. As we saw in the last section, the scaling behavior of the energy

$$\mathcal{E}_0 = \mathcal{J}\big(1 + \frac{a_1^{(0)}}{\mathcal{J}^2} + \frac{a_2^{(0)}}{\mathcal{J}^4} + \dots\big), \tag{5.130}$$

was crucial for the suppression of quantum correction. For generic values of the J_i's, we can only maintain this scaling behavior if we keep γJ fixed as $J \to \infty$ corresponding to a weakly deformed five-sphere. Hence, in order to maintain the semiclassical behavior for general values of the angular momenta we need to take $\gamma \ll 1$.

The expression (5.129) reduces to (5.46) when $J_1 = J_2 = J_3$. Thus, the energy of spinning strings is only affected by the deformation when the angular momenta are "unevenly" distributed on S^5. This suggests another limit where (5.129) is valid. Namely, taking $J_1, J_2, J_3 \to \infty$ and γ, $J_1 - J_2$, $J_2 - J_3$, and $J_1 - J_3$ finite.

6 Outlook

We have verified that the energy of Frolov-Tseytlin strings in Lunin-Maldacena background indeed matches the conformal dimension of scalar gauge theory operators at one-loop. The deformed version of the correspondence thus seems to hold true, extending the gauge/string duality to a sector with less supersymmetry. In fact, there is a certain class of three parameter deformations that breaks all supersymmetry in the gauge theory. The gravity dual of this non-supersymmetric deformation was constructed in [41] and positive tests, similar to what we have done in this thesis, was performed in [41, 33].

The investigation of AdS/CFT duality is a field in rapid development. The correspondence is by now widely believed to be true, but the precise map between string states and gauge theory operators, still remains to be clarified. In this thesis we have matched the energy of Frolov-Tseytlin strings with anomalous dimensions of bosonic operators. Solving the $SU(2)$ Bethe equations, involved a set of M integers, which we assumed to be equal in order to obtain the rational solution corresponding to the Frolov-Tseytlin string. This is at an observational level. We do not know how to recognize the correspondence between these particular solutions a priori. If we, instead of one, had chosen two independent integers the expression for the anomalous dimension would involve elliptic integrals, which can be reproduced as a certain folded string solution in the limit of large angular momentum

[39, 42]. In general, one would expect that different distributions of Bethe roots correspond to different string states. However, the precise mapping is difficult because we are only able to obtain results in the thermodynamic limit.

In the present thesis, we have only discussed the one-loop part of the dilatation operator, which was shown to be integrable in appendix B. Evidence that integrability extends to higher loops was found in [7] and a proposal for a set of all-loop asymptotic Bethe equations was put forward in [43] and soon generalized to the full $PSU(2,2|4)$ symmetry algebra [31]. The S-matrix formalism seems better suited for going beyond one-loop, which in part is why we have chosen to focus on the S-matrix approach, instead of the more rigorous algebraic Bethe ansatz and R-matrix formalism. Indeed, in [44] the complete S-matrix corresponding to the $SU(2|2)$ sector of excitations was constructed (up to an abelian phase), and it was shown to satisfy the Yang-Baxter equation implying all-loop integrability. The All-loop Bethe equations conjectured in [31] can then be derived up to an unknown phase.

The results of [44] allow one to obtain non-perturbative results in the gauge theory, making comparisons with classical string theory possible. This was exploited by Hofman and Maldacena in [45], where the one-magnon state is identified in string theory for strings with infinite angular momentum. Their results have been generalized to certain bound magnon states [46, 47, 48, 49], but it is not yet clear how to identify the general M-magnon state in string theory. Nevertheless, the identification of the "giant magnon" as it has been dubbed, is a major step forward in unraveling the gauge theory/spin chain/string theory duality. If the magnons indeed correspond to fundamental excitations in string theory, one might hope that developing this picture and identifying the string magnons, can lead to a proper quantization of string theory in $AdS_5 \times S^5$.

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A Conformal Coordinate Transformations

In this appendix, we will derive the general form of infinitesimal conformal transformations in d spacetime dimensions. We consider the infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{A.1}$$

and it was shown in (2.48)-(2.50), that the requirement of (A.1) being a conformal transformation leads to the differential equation for $\epsilon(x)$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\sigma \epsilon^\sigma g_{\mu\nu}. \tag{A.2}$$

To solve this, we start by showing that the third derivative of $\epsilon(x)$ vanishes. Contracting both sides of (A.2) with $\partial^\mu \partial^\nu$ gives

$$(1 - \frac{1}{d}) \partial^2 \partial_\mu \epsilon^\mu = 0, \tag{A.3}$$

which for $d > 1$ implies that $\partial^2 \partial_\mu \epsilon^\mu = 0$. Next, we contract (A.2) with $\partial_\rho \partial^\nu$ and get

$$\partial^2 \partial_\rho \epsilon^\mu + (1 - \frac{2}{d}) \partial_\rho \partial_\mu \partial_\sigma \epsilon^\sigma = 0. \tag{A.4}$$

Adding the same equation with μ and ρ interchanged and using (A.2) give

$$\frac{2}{d} \partial^2 \partial_\sigma \epsilon^\sigma g_{\rho\mu} + 2(1 - \frac{2}{d}) \partial_\rho \partial_\mu \partial_\sigma \epsilon^\sigma = 0. \tag{A.5}$$

Using (A.3), this implies that $\partial_\rho \partial_\mu \partial_\sigma \epsilon^\sigma = 0$ for $d > 2$. We now define the tensor $G_{\mu\nu\rho\sigma} = \partial_\mu \partial_\nu \partial_\rho \epsilon_\sigma$, which is clearly symmetric in the first three indices and satisfies $G_{\mu\nu}{}^\rho{}_\rho = 0$ for $d > 2$. Acting on (A.2) with $\partial_\rho \partial_\sigma$, then makes the right side vanish and $G_{\mu\nu\rho\sigma}$ should thus be antisymmetric in the last two indices. However, this contradicts the symmetry in the first three indices as can be seen by writing:

$$G_{\mu\nu\rho\sigma} = G_{\mu\rho\nu\sigma} = -G_{\mu\rho\sigma\nu} = -G_{\mu\sigma\rho\nu} = G_{\mu\sigma\nu\rho} = G_{\mu\nu\sigma\rho}, \tag{A.6}$$

and we conclude that $G_{\mu\nu\rho\sigma} = 0$.

We will now expand ϵ^μ in powers of x, but since the third derivative vanishes, the expansion cannot involve higher powers than x^2. The expansion then becomes

$$\epsilon^\mu(x) = \alpha^\mu + \beta^\mu{}_\nu x^\nu + \gamma^\mu{}_{\nu\rho} x^\nu x^\rho. \tag{A.7}$$

Substituting this into (A.2), and noting that $\gamma_{\mu\nu\rho}$ should be symmetric in the last two indices, gives the equations

$$\beta_{\mu\nu} + \beta_{\nu\mu} = \frac{2}{d} \beta^\rho{}_\rho g_{\mu\nu}, \tag{A.8}$$

$$\gamma_{\mu\nu\rho} + \gamma_{\nu\mu\rho} = \frac{2}{d} \gamma^\sigma{}_{\sigma\rho} g_{\mu\nu}.. \tag{A.9}$$

Equation (A.8) shows that the symmetric part of $\beta_{\mu\nu}$ is simply proportional to $g_{\mu\nu}$ and we know that the antisymmetric part corresponds to the usual Lorentz transformations. To find an expression for $\gamma_{\mu\nu\rho}$, we define $b_\rho = -\frac{1}{d}\gamma^\sigma{}_{\sigma\rho}$ and get from (A.9):

$$\begin{aligned}
\gamma_{\mu\nu\rho} &= -\gamma_{\nu\mu\rho} - 2b_\rho g_{\mu\nu} \\
&= -\gamma_{\nu\rho\mu} - 2b_\rho g_{\mu\nu} \\
&= -2b_\rho g_{\mu\nu} + 2b_\mu g_{\nu\rho} + \gamma_{\rho\nu\mu} \\
&= -2b_\rho g_{\mu\nu} + 2b_\mu g_{\nu\rho} + \gamma_{\rho\mu\nu} \\
&= -\gamma_{\mu\nu\rho} - 2b_\rho g_{\mu\nu} + 2b_\mu g_{\nu\rho} - 2b_\nu g_{\mu\rho}.
\end{aligned}$$

Thus

$$\gamma_{\mu\nu\rho} = b_\mu g_{\nu\rho} - b_\rho g_{\mu\nu} - b_\nu g_{\mu\rho}, \tag{A.10}$$

and the general expression for $\epsilon^\mu(x)$ becomes

$$\epsilon^\mu(x) = \alpha^\mu + \sigma x^\mu + \omega^\mu{}_\nu x^\nu + b^\mu x^2 - 2b_\nu x^\nu x^\mu, \tag{A.11}$$

where we have defined $\sigma = \frac{1}{d}\beta^\rho{}_\rho$. We can now identify the different terms appearing here: α^μ corresponds to spacetime translations, the σ term corresponds to dilatations, the $\omega^\mu{}_\nu$ term corresponds to Lorentz transformations, and the terms involving b^μ are special conformal transformations.

B The Algebraic Bethe Ansatz

In this appendix, we will prove the integrability of the $SU(2)$ spin chain and present an alternative derivation of the Bethe equations. It is a purely algebraic method, which is completely different from the coordinate space ansatz originally introduced by Bethe. We closely follow the review by Faddeev [29].

The method can be regarded as a generalization of the algebraic approach to the harmonic oscillator, where one uses the commutator of raising and lowering operators to obtain the spectrum. The fundamental object in our approach is the Lax operator, which can be used to construct operators corresponding to $\hat{a}^\dagger$ and $\hat{a}$. The commutator $[\hat{a}, \hat{a}^\dagger] = 1$ is now replaced by a Yang-Baxter equation, which is used to obtain the Bethe equations determining the spectrum of the spin chain.

B.1 The Lax Operator and Integrability

A spin chain of length J is a quantum system defined on a Hilbert space, which is the tensor product of local two-dimensional complex vector spaces h_n. The full Hilbert space is thus

$$\mathcal{H} = h_1 \otimes h_2 \otimes \ldots \otimes h_J. \tag{B.1}$$

In each h_n, we will use the ordered basis $\{|\uparrow\rangle, |\downarrow\rangle\}_n$, which are eigenstates of the third component of the spin operator S^3_n with eigenvalues $\{\frac{1}{2}, -\frac{1}{2}\}$.

The hamiltonian of the one-dimensional Heisenberg model with nearest neighbor interaction is given by

$$H_{Sc} = \frac{\varepsilon_0}{2} \sum_{n=1}^{J} (1 - \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}) = \varepsilon_0 \sum_{n=1}^{J} (1 - P_{n,n+1}), \qquad P_{J,J+1} = P_{1,J} \tag{B.2}$$

where $P_{n,n+1}$ is the permutation operator acting on $h_n \otimes h_{n+1}$, and $\boldsymbol{\sigma}_n = 2\mathbf{S}_n$ are the Pauli matrices acting on h_n. Our task is to diagonalize this hamiltonian. For a given J, it reduces to the diagonalization of a $2^J \times 2^J$ matrix accessible by numerical methods, but we would like to obtain eigenvalues, in which we explicitly see the dependence on J. Furthermore, we will be interested in the limit of $J \to \infty$ where only analytical methods work.

We now introduce the Lax operator. This operator will be used to explicitly construct a set of J commuting operators on the spin chain. The hamiltonian belongs to this set and the spectrum is obtained by simultaneously diagonalizing all these operators. The Lax operator $L_{n,a}(u)$ depends on a complex spectral parameter u and acts on the product of an auxiliary vector space V and the local vector space h_n. V is also a two-dimensional complex vector space, and the Lax operator is defined by

$$L_{n,a}(u) = u + i\mathbf{S}_n \cdot \boldsymbol{\sigma}_a = \begin{pmatrix} u + \frac{i}{2}\sigma^3_n & \frac{i}{2}\sigma^- \\ \frac{i}{2}\sigma^+ & u - \frac{i}{2}\sigma^3_n \end{pmatrix}, \tag{B.3}$$

where $\mathbf{S}_n$ acts in h_n and $\boldsymbol{\sigma}_a$ acts in V. In the last equality, we have written the operator as a matrix in V with respect to the ordered basis $\{|\uparrow\rangle, |\downarrow\rangle\}_a$, and the entries are acting in h_n. It will also be convenient to write it in terms of the permutation operator:

$$L_{n,a} = u - \frac{i}{2} + iP_{n,a} \tag{B.4}$$

which follows from the identity $2P_{m,n} = 1 + \boldsymbol{\sigma}_m \cdot \boldsymbol{\sigma}_n$. The Lax operator seems to be similar in structure to the individual terms of the hamiltonian, but it acts in a different vector space. Each term in the hamiltonian acts on adjacent sites on the spin chain, whereas the Lax operator acts on one site on the spin chain and on the auxiliary space V.

Consider now two copies of the auxiliary space V_1 and V_2 and two Lax operators $L_{n,a_1}(u)$ and $L_{n,a_2}(v)$ acting on these two spaces in addition to h_n. We are interested in the commutator of these two operators, but since there is now two auxiliary spaces involved, we have to write the Lax operators as 4×4 matrices with entries in h_n. It turns out that $L_{n,a_1}(u)$ and $L_{n,a_2}(v)$ do not commute, but $L_{n,a_1}(u)L_{n,a_2}(v)$ and $L_{n,a_2}(v)L_{n,a_1}(u)$ are similar operators. This means that an operator, $R_{a_1,a_2}(u-v)$ called the R-matrix defined on $V_1 \otimes V_2$, exists and has the property

$$R_{a_1,a_2}(u-v)L_{n,a_1}(u)L_{n,a_2}(v) = L_{n,a_2}(v)L_{n,a_1}(u)R_{a_1,a_2}(u-v). \tag{B.5}$$

The explicit expression for the R-matrix is

$$R_{a_1,a_2}(u) = u + iP_{a_1,a_2}, \tag{B.6}$$

and we see that it is essentially a Lax operator ($L_{a_1,a_2}(u) = R_{a_1,a_2}(u - i/2)$). Using this expression, it is easy to prove (B.5) if we note that the permutation operators satisfy the relations:

$$P_{n,a_1} P_{n,a_2} = P_{a_1,a_2} P_{n,a_1} = P_{n,a_2} P_{a_1,a_2}. \tag{B.7}$$

We now define the monodromy on the spin chain which is given by

$$T_a(u) = L_{J,a}(u) L_{J-1,a}(u) \dots L_{1,a}(u), \tag{B.8}$$

and acts on the product space $V \otimes \mathcal{H}$. We can also write it as a matrix in the auxiliary space V with entries being operators in $\mathcal{H}$:

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{B.9}$$

Taking the trace in auxiliary space, yields the operator

$$F(u) \equiv tr_a T(u) = A(u) + D(u), \tag{B.10}$$

which is defined on $\mathcal{H}$. In the following, we will show that the hamiltonian can be constructed from $F(u)$ and its eigenstates are obtained by acting with $B(u)$ a number of times on a properly defined groundstate. If we compare with the harmonic oscillator, $B(u)$ and $C(u)$ correspond to the raising and lowering operators, respectively, and the total spin will play the role of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$.

Referring to (B.4), we see that the monodromy is a polynomial in u of order J:

$$T_a(u) = u^J + i u^{J-1} \sum_{n=1}^{J} \mathbf{S}_n \cdot \boldsymbol{\sigma}_a + \dots, \tag{B.11}$$

implying that $F(u)$ has a similar non-trivial expansion in u

$$F(u) = 2u^J + \sum_{n=0}^{J-2} Q_n u^n. \tag{B.12}$$

Note that Q_{J-1} vanishes since it is a sum of terms involving the trace $tr_a[\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_a] = \sigma^3 - \sigma^3 = 0$. It will now be shown that the operators Q_n all commute among themselves. If we augment this set by the total spin operator S^3, which is shown to commute with $F(u)$ below, we have a set of J commuting operators, proving the integrability of the spin chain. We will also show how the hamiltonian can be constructed from this family of operators.

First we need to establish the commutation relations for $T_a(u)$. We will consider two copies of the monodromy acting in different auxiliary spaces V_1 and V_2. The result involves the R-matrix (B.6) and has the exact same form as (B.5):

$$R_{a_1,a_2}(u - v) T_{a_1}(u) T_{a_2}(v) = T_{a_2}(v) T_{a_1}(u) R_{a_1,a_2}(u - v). \tag{B.13}$$

This result is easily derived using (B.5) and the definition of the monodromy:

$$
\begin{aligned}
R_{a_1,a_2}(u-v)T_{a_1}(u)T_{a_2}(v) &= R_{a_1,a_2}(u-v)L_{J,a_1}(u)\dots L_{1,a_1}(u)L_{J,a_2}(v)\dots L_{1,a_2}(v)\\
&= R_{a_1,a_2}(u-v)L_{J,a_1}(u)L_{J,a_2}(v)\dots L_{1,a_1}(u)L_{1,a_2}(v)\\
&= L_{J,a_2}(v)L_{J,a_1}(u)\dots L_{1,a_2}(v)L_{1,a_1}(u)R_{a_1,a_2}(u-v)\\
&= L_{J,a_2}(v)\dots L_{1,a_2}(v)L_{J,a_1}(u)\dots L_{1,a_1}(u)R_{a_1,a_2}(u-v)\\
&= T_{a_2}(v)T_{a_1}(u)R_{a_1,a_2}(u-v),
\end{aligned}
$$

where in the second and fourth lines, we used that Lax operators acting in different vector spaces commute, and in the third line, we used (B.5) J times. This equation involves 4×4 matrices acting in $V_1\otimes V_2$ with entries being operators in $\mathcal{H}$, and the expression is thus a compact way of writing 16 commutation relations for $A(u)$, $B(u)$, $C(u)$, and $D(u)$. It is useful to explicitly write (B.13) as a matrix equation. Defining the ordered basis in $V_1\otimes V_2$

$$
\{|\uparrow\rangle_1\otimes|\uparrow\rangle_2, |\downarrow\rangle_1\otimes|\uparrow\rangle_2, |\uparrow\rangle_1\otimes|\downarrow\rangle_2, |\downarrow\rangle_1\otimes|\downarrow\rangle_2\}, \tag{B.14}
$$

we see that the two copies of the monodromy can be written

$$
T_{a_1}(u)=\begin{pmatrix} A & B & 0 & 0\\ C & D & 0 & 0\\ 0 & 0 & A & B\\ 0 & 0 & C & D\end{pmatrix}, \qquad T_{a_2}(v)=\begin{pmatrix} A' & 0 & B' & 0\\ 0 & A' & 0 & B'\\ C' & 0 & D' & 0\\ 0 & C' & 0 & D'\end{pmatrix}, \tag{B.15}
$$

with $A=A(u)$, $A'=A(v)$ and so on. The R-matrix can be written

$$
R_{a_1,a_2}(u)=\begin{pmatrix} u+i & 0 & 0 & 0\\ 0 & u & i & 0\\ 0 & i & u & 0\\ 0 & 0 & 0 & u+i\end{pmatrix}, \tag{B.16}
$$

and (B.13) becomes

$$
\begin{aligned}
&\begin{pmatrix} (i+w)AA' & (i+w)BA' & (i+w)AB' & (i+w)BB'\\ iAC'+wCA' & iBC'+wDA' & iAD'+wCB' & iBD'+wDB'\\ iCA'+wAC' & iDA'+wBC' & iCB'+wAD' & iDB'+wBD'\\ (i+w)CC' & (i+w)DC' & (i+w)CD' & (i+w)DD'\end{pmatrix} =\\
&\begin{pmatrix} (i+w)A'A & iB'A+wA'B & iA'B+wB'A & (i+w)B'B\\ (i+w)A'C & iB'C+wA'D & iA'D+wB'C & (i+w)B'D\\ (i+w)C'A & iD'A+wC'B & iC'B+wD'A & (i+w)D'B\\ (i+w)C'C & iD'C+wC'D & iC'D+wD'C & (i+w)D'D\end{pmatrix},
\end{aligned} \tag{B.17}
$$

where $w=u-v$. First, we observe that the four entries of the monodromy commute among themselves:

$$
[A,A']=[B,B']=[C,C']=[D,D']=0. \tag{B.18}
$$

Furthermore, equating the sum of diagonal elements gives the equation

$$(u-v)\big([A,D']+[D,A']\big)=i\big([B',C]+[C',B]\big). \tag{B.19}$$

We then note that the left hand side is invariant under exchange of u and v, whereas the right hand side acquires a minus under the exchange. This implies that the two expressions vanish.

The commutator relations show that the trace of the monodromy commute at different values of the spectral parameter:

$$[F(u),F(v)]=0, \tag{B.20}$$

and this in turn implies that all the Q_n in (B.12) commute, since we can write

$$[Q_m,Q_n]=\frac{1}{m!n!}\frac{d^m}{du^m}\frac{d^n}{dv^n}[F(u),F(v)]\Big|_{u=v=0}=0. \tag{B.21}$$

We will now see how to derive the hamiltonian from $F(u)$. The Lax operator becomes particularly simple when evaluated at $u=i/2$ and so does the monodromy and $F(u)$. We have that

$$\begin{aligned}T_a(i/2)&=i^J P_{J,a}P_{J-1,a}\dots P_{1,a}\\&=i^J P_{1,2}P_{2,3}\dots P_{J-1,J}P_{J,a},\end{aligned} \tag{B.22}$$

where the last equality follows from repeated use of the relations (B.7). For example, we can write $P_{2,a}P_{1,a}=P_{1,2}P_{2,a}$ and then bring $P_{1,2}$ all the way through the string of remaining permutations, which do not act in h_1 or h_2. Taking the trace in auxiliary space, only involves $P_{J,a}$, and using that $P_{J,a}=\frac{1}{2}(1+\boldsymbol{\sigma}_J\cdot\boldsymbol{\sigma}_a)$, one can easily check that $tr_aP_{J,a}=1$. We can now define the shift operator

$$U=i^{-J}F(i/2)=i^{-J}tr_aT(i/2)=P_{1,2}P_{2,3}\dots P_{J-1,J}, \tag{B.23}$$

which is seen to shift variables on site n to $n+1$. Since $P^\dagger=P^{-1}=P$, we immediately observe that U is unitary

$$UU^\dagger=U^\dagger U=1, \tag{B.24}$$

and local operators will transform as $UX_nU^{-1}=X_{n+1}$. As usual, the shift operator allows us to introduce a hermitian momentum operator P which acts as the generator of infinitesimal shifts. On a lattice, this corresponds to a shift along one site, so we define it by

$$U=e^{iP}. \tag{B.25}$$

Having examined $F(i/2)$, we take the next step and analyze the first derivative of F evaluated at $i/2$. We start by calculating the derivative of $T_a(u)$ and evaluate it at $i/2$.

To do this, it is most convenient to write $T_a(u)$ as a string of Lax operators using the expression (B.4). It is then easy to see that

$$\frac{d}{du}T_a(u)\Big|_{u=i/2} = i^{J-1}\sum_{n=1}^{J} P_{J,a}\dots\widehat{P_{n,a}}\dots P_{1,a}, \tag{B.26}$$

where $\widehat{\ }$ means that the factor is absent. Rearranging the string of permutations as in (B.22), we can calculate the trace

$$\frac{d}{du}F(u)\Big|_{u=i/2} = i^{J-1}\sum_{n=1}^{J} P_{1,2}\dots P_{n-1,n+1}\dots P_{J-1,J}. \tag{B.27}$$

Multiplying from the left with $i^{1-J}U^{-1}$ and using that $P_{i,j}^2 = 1$ give

$$i^{1-J}\frac{d}{du}F(u)\Big|_{u=i/2}U^{-1} = \sum_{n=1}^{J} P_{n,n+1}. \tag{B.28}$$

Comparing with (B.2), we see that the hamiltonian can be written

$$H = \varepsilon_0\Big(J - i\frac{d}{du}\ln F(u)\Big|_{u=i/2}\Big), \tag{B.29}$$

where we used that $U = i^{-J}F(i/2)$.

B.2 Spectrum and Bethe Equations

We have just shown that the total momentum and the hamiltonian belong to the family of commuting operators generated by $F(u)$. In this subsection, we diagonalize $F(u)$ and thus the whole set of commuting observables simultaneously. In particular we will get an expression for the eigenvalues of the hamiltonian in terms of certain complex variables - the Bethe roots.

As already mentioned, the algebraic method is a generalization of the method used to diagonalize the hamiltonian of the harmonic oscillator. Remember that one diagonalizes the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ using the commutator relations $\hat{n}\hat{a} = \hat{a}(\hat{n}-1)$, $\hat{n}\hat{a}^\dagger = \hat{a}^\dagger(\hat{n}+1)$ and defining a "highest weight state", which is annihilated by $\hat{a}$.

We define the state with highest weight in h_n to be $|\uparrow\rangle_n$. This state has the property that it is an eigenstate of σ^3 and is annihilated by σ^+. The Lax operator thus takes a simple triangular form in auxiliary space when applied to it:

$$L_n(u)\begin{pmatrix} a \\ b \end{pmatrix}|\uparrow\rangle_n = \begin{pmatrix} u+i/2 & iS_n^- \\ 0 & u-i/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}|\uparrow\rangle_n, \tag{B.30}$$

where a and b is the coordinate representation of a state in V. Defining a highest weight state in $\mathcal{H}$ by

$$|\Omega\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \dots \otimes |\uparrow\rangle_J, \tag{B.31}$$

we see that the monodromy becomes triangular in the auxiliary space when acting on this state:

$$T(u)\begin{pmatrix} a \\ b \end{pmatrix}|\Omega\rangle_n = \begin{pmatrix} (u+i/2)^J & B(u) \\ 0 & (u-i/2)^J \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}|\Omega\rangle_n. \tag{B.32}$$

and $B(u)$ becomes a linear combination of S^-'s. This shows that $|\Omega\rangle$ is an eigenstate of A and D and is annihilated by C:

$$C(u)|\Omega\rangle = 0, \qquad A(u)|\Omega\rangle = (u+i/2)^J|\Omega\rangle, \qquad D(u)|\Omega\rangle = (u-i/2)^J|\Omega\rangle. \tag{B.33}$$

The operator $C(u)$ is thus playing the role of $\hat{a}$ annihilating the highest weight state. We defined $|\Omega\rangle$ as an eigenstate of the third component of the total momentum $S^3 = \sum_{n=1}^J S_n^3$ with maximum eigenvalue $J/2$, and we will now show that we obtain new eigenstates of S^3 by acting on $|\Omega\rangle$ with $B(u)$.[19]

To see this, we return to the commutation relations (B.13) and consider the limit $v \to \infty$. Keeping only the leading v^{J+1} terms gives a trivial result so we also need to keep terms of order v^J. Using (B.11), the resulting equation is

$$(u + iP_{a_1,a_2})T_{a_1}(u) - iT_{a_1}(u)\mathbf{S}\cdot\boldsymbol{\sigma}_{a_2} = T_{a_1}(u)(u+iP_{a_1,a_2}) - i\mathbf{S}\cdot\boldsymbol{\sigma}_{a_2}T_{a_1}(u), \tag{B.34}$$

or

$$[P_{a_1,a_2} + \mathbf{S}\cdot\boldsymbol{\sigma}_{a_2}, T_{a_1}(u)] = 0, \tag{B.35}$$

where $\mathbf{S} = \sum_{n=1}^J \mathbf{S}_n$ is the total spin. We should again be a bit careful when evaluating these products. Since there is two auxiliary spaces, we should represent the monodromy and $\boldsymbol{\sigma}_{a_2}$ by 4×4 matrices. Using the ordered basis (B.14), the matrices in $V_1 \otimes V_2$ become

$$P_{a_1,a_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_{a_1}(u) = \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{pmatrix},$$
$$\mathbf{S}\cdot\boldsymbol{\sigma}_{a_2} = \begin{pmatrix} S^3 & 0 & S^- & 0 \\ 0 & S^3 & 0 & S^- \\ S^+ & 0 & -S^3 & 0 \\ 0 & S^+ & 0 & -S^3 \end{pmatrix}.$$

The commutator relations contained in (B.35) can then be extracted:

$$[S^3, A] = [S^3, D] = [S^-, B] = [S^+, C] = 0, \tag{B.36}$$
$$[S^+, B] = -[S^-, C] = A - D, \tag{B.37}$$
$$[S^3, B] = -B, \quad [S^3, C] = C. \tag{B.38}$$

[19] We could as well have picked the state with all down spins as a reference state. Then the roles of B and C would be exchanged.

The first line shows that $F(u)$ and S^3 commute and last line shows that B and C indeed lowers and raises the eigenvalue of S^3 by 1, respectively.

We are thus led to consider the states

$$|M, \{u_j\}\rangle = B(u_1)B(u_2)\dots B(u_M)|\Omega\rangle, \tag{B.39}$$

which are eigenstates of S^3 with eigenvalue $J/2 - M$. In fact, these states are also eigenvalues of $F(u)$ as we will now show. Using the matrix form of the commutator relations (B.13), we obtain the expressions:

$$A(u)B(v) = f(u-v)B(v)A(u) + g(u-v)B(u)A(v), \tag{B.40}$$
$$D(u)B(v) = h(u-v)B(v)D(u) - g(u-v)B(u)D(v), \tag{B.41}$$

with

$$f(u) \equiv \frac{u-i}{u}, \quad g(u) \equiv \frac{i}{u}, \quad h(u) \equiv \frac{u+i}{u}. \tag{B.42}$$

The last term in (B.40) shows that $|M, \{u_j\}\rangle$ is not an eigenstate of $A(u)$. We get 2^M terms, whereas only one is proportional to $|M, \{u_j\}\rangle$. The remaining terms are obtained by moving A through the string of $B's$. At each B, we either let A pass freely picking up a factor of f, or exchange variables picking up a factor of g. Since the B's commute, we can group all these terms into M terms, where A at the end of the string depends on a certain u_k. The result can be written

$$\begin{aligned} A(u)|M, \{u_j\}\rangle =& \prod_{j=1}^{M} f(u-u_j)(u+i/2)^J |M, \{u_j\}\rangle \\ &+ \sum_{k=1}^{M} \alpha_k(u, \{u_j\}) B(u_1)\dots\widehat{B(u_k)}\dots B(u_M)B(u)|\Omega\rangle. \end{aligned} \tag{B.43}$$

To determine the coefficient functions α_k, we start by noting that the term involving α_1 results from moving A through the string of B's exchanging variables only with the first factor of B. The result is thus

$$\alpha_1(u, \{u_j\}) = g(u-u_1) \prod_{j=2}^{M} f(u_1 - u_j)(u_1 + i/2)^M. \tag{B.44}$$

Since the B's commute, we can move any $B(u_k)$ to the first position in the string of B's in $|M, \{u\}\rangle$ and using the argument above, thus yields

$$\alpha_k(u, \{u_j\}) = g(u-u_k) \prod_{j\neq k}^{M} f(u_k - u_j)(u_k + i/2)^M. \tag{B.45}$$

Using (B.41), we can also apply this arguments for the action of $D(u)$ on $|M, \{u_j\}\rangle$. We get

$$D(u)|M,\{u_j\}\rangle = \prod_{j=1}^{M} h(u-u_j)(u-i/2)^J |M,\{u_j\}\rangle + \sum_{k=1}^{M} \beta_k(u,\{u_j\}) B(u_1)\dots\widehat{B(u_k)}\dots B(u_M)B(u)|\Omega\rangle. \tag{B.46}$$

with

$$\beta_k(u,\{u_j\}) = -g(u-u_k)\prod_{j\neq k}^{M} h(u_k-u_j)(u_k-i/2)^M. \tag{B.47}$$

We now require that the state $|M, \{u_j\}\rangle$ is an eigenstate of $F(u)$. This can only be true if the second line in (B.43) cancel against the second line of (B.46). Thus, if the set $\{u_j\}$ satisfies

$$(u_k+i/2)^J \prod_{j\neq k}^{M} f(u_k-u_j) = (u_k-i/2)^J \prod_{j\neq k}^{M} h(u_k-u_j), \tag{B.48}$$

we get that

$$F(u)|M,\{u_j\}\rangle = \Lambda(u,\{u_j\})|M,\{u_j\}\rangle, \tag{B.49}$$

with

$$\Lambda(u,\{u_j\}) = (u+i/2)^J \prod_{j=1}^{M} f(u-u_j) + (u-i/2)^J \prod_{j=1}^{M} h(u-u_j). \tag{B.50}$$

The M equations (B.48) are called the Bethe equations and the complex spectral parameters u_j are called Bethe roots. We can write the Bethe equations in a more compact form using (B.42),

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^J = \prod_{j\neq k}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}, \tag{B.51}$$

which is the familiar form derived from the coordinate space Bethe ansatz in (4.16).

We will now derive the eigenvalues of the hamiltonian and momentum operators. Since the states $|M, \{u_j\}\rangle$ with $\{u_j\}$ satisfying the Bethe equations are eigenstates of $F(u)$, they are eigenstates of momentum and energy. Starting with momentum, we note that $\Lambda(u, \{u_j\})$ becomes particularly simple when evaluated at $i/2$, giving

$$e^{iP}|M,\{u_j\}\rangle = i^{-J}\Lambda(i/2,\{u_j\})|M,\{u_j\}\rangle = \prod_{j=1}^{M} \frac{u_j+i/2}{u_j-i/2}|M,\{u_j\}\rangle. \tag{B.52}$$

The total momentum is thus additive with

$$P = \sum_{j=1}^{M} p_j, \qquad p_j = -i \ln \frac{u_j + i/2}{u_j - i/2}. \tag{B.53}$$

The same is true for the energy, and using (B.29), we get

$$H|M, \{u_j\}\rangle = \varepsilon_0 \Big(J - i \frac{d}{du} \ln \Lambda(u, \{u_j\}) \Big|_{u=i/2} \Big) |M, \{u_j\}\rangle \tag{B.54}$$

giving

$$E = \varepsilon_0 \sum_{j=1}^{M} \epsilon_j, \qquad \epsilon_j = \frac{1}{u_j^2 + 1/4}. \tag{B.55}$$

The fact that energy and momentum are additive allows us to interpret the operators $B(u_j)$ as creation operators creating particles of momentum p_j and energy ϵ_j. The Bethe equations are constraints on the momenta, which have to be satisfied for the states to be eigenstates of the hamiltonian. We also note that the unitary shift operator allowed us to define a hermitian momentum operator with real eigenvalues. The individual momentum variables p_j however, can be complex numbers giving rise to bound states.

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